SCM 228 TUTORIAL 6

LAPLACE TRANSFORMS

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

where $F(s) = \mathcal{L}(f(t))$.

$$\mathcal{L}\{f * g\} = \mathcal{L}(f(t))\,\mathcal{L}(g(t)) = F(s)G(s)$$

where $f * g = \int_0^t f(T)\, g(t-T)\, dT$

EXAMPLES

1. Z and C, 4.4 №1 (p.222)

$\mathcal{L}\{t \cos 2t\}$.

Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$,

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -\frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}.$$

2. Z and C, 4.4 №15 (p.222)

$$\mathcal{L}\left\{\int_0^t T e^{t-T}\, dT\right\}$$

$$= \mathcal{L}\{t * e^t\}$$

$$= \frac{1}{s^2} \times \frac{1}{s-1} = \frac{1}{s^2(s-1)}$$

NOTE: If convolutions are not used, $\int_0^t T e^{t-T}\, dT$

$$= e^t \int_0^t T e^{-T}\, dT$$

$$= e^t \left[-(1+T)e^{-T}\right]_0^t$$

(Schaum Table of Integrals)

$$= e^t\left[1-(1+t)e^{-t}\right]$$

$$= e^t - 1 - t.$$

$$\therefore \mathcal{L}\left\{\int_0^t T e^{t-T} dT\right\} = \mathcal{L}\{e^t - 1 - t\}$$

$$= \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$$

$$= \frac{s^2 - s^2 + s - s + 1}{s^2(s-1)} = \frac{1}{s^2(s-1)}.$$

3. Solve

$$(x^2+2)y'' + 3xy' - y = 0 \quad ; \quad (1)$$

$$y(0) = 1, \quad y'(0) = 2.$$

Let $y = \sum_0^\infty a_n x^n$

$\therefore y' = \sum_0^\infty n a_n x^{n-1}$;

$y'' = \sum_0^\infty n(n-1) a_n x^{n-2}$.

Sub. in (1)

$$\therefore (x^2+2)\sum_0^\infty n(n-1)a_n x^{n-2} + 3x\sum_0^\infty n a_n x^{n-1} - \sum_0^\infty a_n x^n = 0.$$

$$\therefore \sum_0^\infty \left[n(n-1)a_n x^n + 2n(n-1)a_n x^{n-2} + 3n a_n x^n - a_n x^n\right] = 0$$

As $n(n-1) + 3n - 1$

$= n^2 + 2n - 1$,

$$\sum_0^\infty\left[(n^2+2n-1)a_n x^n + 2n(n-1)a_n x^{n-2}\right] = 0$$

Equate co-eff$^{\underline{ts}}$ of x^m

($n = m$ in 1st term ; $n-2 = m$ in 2nd)

$$\therefore (m^2+2m-1)a_m + 2(m+2)(m+1)a_{m+2} = 0$$

$$\therefore a_{m+2} = \frac{-(m^2+2m-1)}{2(m+1)(m+2)} a_m.$$

As $y(0) = 1$, $a_0 = 1$.

As $y'(0) = 2$, $a_1 = 2$.

$\therefore a_2 = \frac{1}{4} a_0 = \frac{1}{4}$

$a_4 = -\frac{7}{24} a_2 = -\frac{7}{96}$, etc.

$\therefore a_3 = \frac{-2}{12} a_1 = -\frac{1}{3}$.

$a_5 = \frac{-14}{40} a_3 = \frac{7}{60}$ etc.

$$\therefore y = 1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \dots$$

$$+ 2x - \frac{1}{3}x^3 + \frac{7}{60}x^5 + \dots$$

EXERCISES:

1. Find $\mathcal{L}\{t \cosh 4t\}$

$$= -\frac{d}{ds}\left(\frac{s}{s^2-16}\right) = \frac{-(s^2-16-2s^2)}{(s^2-16)^2}$$

$$= \frac{s^2+16}{(s^2-16)^2}$$

2. Find $\mathcal{L}\{t * \sin t\}$

$$= \mathcal{L}\{t\}\,\mathcal{L}\{\sin t\}$$

$$= \frac{1}{s^2} \cdot \frac{1}{s^2+1} = \frac{1}{s^2(s^2+1)}.$$