WAVE EQUATION with fixed ends and $u(x,0)=f(x)$; $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ has solution

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi at}{L}+B_n\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}$$

with $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

and $B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$

LAPLACE'S EQUATION with insulated sides ($x=0$ and $x=a$), and $u(x,0)=0$, $u(x,b)=f(x)$ has solution

$$u(x,y)=A_0 y+\sum_{n=1}^{\infty}A_n\sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}$$

with $A_0 = \frac{1}{ab}\int_0^a f(x)\,dx$

and $A_n = \frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$

EXAMPLES

1. Z and C. 13·4 № 1, p. 702.

Here $f(x)=\frac{1}{4}x(L-x)$ and $g(x)=0$.

$\therefore B_n = 0$.

$$A_n = \frac{2}{L}\int_0^L \frac{1}{4}x(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2L}\int_0^L (Lx-x^2)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2L}\left\{L\int_0^L x\sin\frac{n\pi x}{L}\,dx - \int_0^L x^2\sin\frac{n\pi x}{L}\,dx\right\}$$

$$\therefore A_n = \frac{1}{2L}\left\{L\left[\frac{-xL\cos\frac{n\pi x}{L}}{n\pi}+\frac{L^2\sin\frac{n\pi x}{L}}{n^2\pi^2}\right]_0^L - \left[\frac{-Lx^2\cos\frac{n\pi x}{L}}{n\pi}+\frac{2L^2x\sin\frac{n\pi x}{L}}{n^2\pi^2}+\frac{2L^3\cos\frac{n\pi x}{L}}{n^3\pi^3}\right]_0^L\right\}$$

$$\therefore A_n = \frac{1}{2L}\left\{\frac{-L^3}{n\pi}\cos n\pi+\frac{L^3}{n\pi}\cos n\pi - \frac{2L^3}{n^3\pi^3}\cos n\pi + \frac{2L^3}{n^3\pi^3}\right\}$$

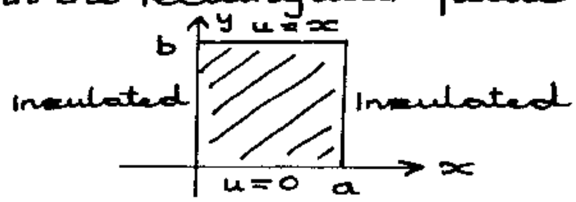
$$\therefore A_n = \frac{1}{2L}\cdot\frac{2L^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore A_n = \frac{L^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$\therefore$ The solution is

$$u(x,t)=\frac{L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\cos\frac{n\pi at}{L}\cdot\sin\frac{n\pi x}{L}$$

2. Find the temperature $u(x,y)$ in the rectangular plate



Since $u(x,b)=f(x)=x$,

$$A_0 = \frac{1}{ab}\int_0^a x\,dx = \frac{1}{ab}\left[\tfrac{1}{2}x^2\right]_0^a$$

$\therefore A_0 = \frac{a}{2b}$.

Also $\int_0^a x\cos\frac{n\pi x}{a}\,dx$

$$= \left[\frac{ax\sin\frac{n\pi x}{a}}{n\pi}+\frac{a^2\cos\frac{n\pi x}{a}}{n^2\pi^2}\right]_0^a$$

$$= \left(0+\frac{a^2\cos n\pi}{n^2\pi^2}\right)-\left(0+\frac{a^2}{n^2\pi^2}\right)$$

$$= \frac{a^2}{n^2\pi^2}\left[(-1)^n-1\right].$$

$$\therefore A_n = \frac{2a}{\sinh\frac{n\pi b}{a}}\left[(-1)^n-1\right]\Big/n^2\pi^2$$

$$\therefore u(x,y) = \frac{ay}{2b} + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}$$

EXERCISE. In example 1, find the solution when $g(x)=1$.

$$\therefore B_n = \frac{2}{n\pi a}\int_0^L \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-2}{n\pi a}\cdot\frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{-2L}{an^2\pi^2}\left[(-1)^n-1\right]$$

$$\therefore B_n = \frac{2L}{an^2\pi^2}\left[1-(-1)^n\right].$$