SCM 228 TUTORIAL 9

SEPARABLE PARTIAL D.E's

ONE-DIMENSIONAL HEAT EQU^N.

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (k>0)$$

ONE-DIMENSIONAL WAVE EQU^N.

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

TWO-DIMENSIONAL LAPLACE EQU^N.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

EXAMPLES

1. Find a product solution to $4\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial y} = 0$ using separation of variables

Let $u(x,y) = X(x)Y(y)$

$\therefore \frac{\partial u}{\partial x} = Y\frac{dX}{dx}\ ;\ \frac{\partial u}{\partial y} = X\frac{dY}{dy}$

$\therefore 4Y\frac{dX}{dx} = 5X\frac{dY}{dy}$

$(\div XY)\ \therefore \frac{4}{X}\frac{dX}{dx} = \frac{5}{Y}\frac{dY}{dy} = k$

So, $\frac{4}{X}\frac{dX}{dx} = k$

$\therefore \int\frac{4}{X}dX = \int k\,dx$

$\therefore 4\ln|X| = kx + C_1$

$\therefore |X| = 1\,e^{(kx+c_1)/4}$

$\therefore X = Ae^{kx/4}$

similarly, $Y = Be^{ky/5}$

$\therefore u(x,y) = ABe^{k(x/4+y/5)}$

$\therefore u(x,y) = Ce^{k(x/4+y/5)}$

$\left(= Ce^{\lambda(5x+4y)}\right)$.

2. Zand C. 13.3 № 3. p.699

$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\ ;\ 0<x<L,\ t>0$

Insulated ends means

$u_x(0,t) = 0\ ;\ u_x(L,t) = 0$

Also $u(x,0) = f(x)\ ;\ 0<x<L$

Letting $u(x,t) = X(x)T(t)$

$\frac{\partial^2 u}{\partial x^2} = TX''\ ;\ \frac{\partial u}{\partial t} = X\dot{T}$

$\therefore$ The equ^n becomes

$kTX'' = X\dot{T}$

$(\div kXT)\ \therefore \frac{X''}{X} = \frac{\dot{T}}{kT} = -\lambda^2$.

$\therefore\ X'' + \lambda^2 X = 0 \quad (1)$

$\dot{T} + \lambda^2 kT = 0 \quad (2)$

The boundary conditions give

$X'(0) = X'(L) = 0. \quad (3)$

(i) For $\lambda = 0$, (1) has solution $X(x) = ax+b$

From (3), $a = 0 \quad \therefore X(x) = b \quad (4)$

(ii) For $\lambda^2>0$, (1) has solution

$X = \tilde{a}\cos\lambda x + \tilde{b}\sin\lambda x$

$\therefore X' = \lambda(-\tilde{a}\sin\lambda x + \tilde{b}\cos\lambda x)$

From (3), $\tilde{b} = 0$ and $\sin\lambda L = 0$

$\therefore \lambda L = n\pi$ so $\lambda = \frac{n\pi}{L}$.

$\therefore X(x) = \tilde{a}_n\cos\left(\frac{n\pi x}{L}\right). \quad (5)$

For $\lambda = 0$, (2) becomes $\dot{T} = 0$

$\therefore T = c. \quad (6)$

$\therefore$ 1 sol^n is $u = bc = a_0/2 \quad (7)$

For $\lambda = \frac{n\pi}{L}$, (2) becomes

$\dot{T} + \frac{n^2\pi^2}{L}T = 0$

$\therefore T = c_n e^{-n^2\pi^2 t/L}. \quad (8)$

By superposition

$u = \sum_{n=1}^{\infty}\tilde{a}_n c_n e^{-n^2\pi^2 t/L}\cos\left(\frac{n\pi x}{L}\right)$

is another sol^n.

$\therefore u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/L}\cos\left(\frac{n\pi x}{L}\right) \quad (9)$

Now $u(x,0) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$

i.e. $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$\therefore$ The sol^n is (9) with a_0 and a_n as above.

EXERCISE. Solve example 2, when $f(x) = x\ ;\ 0<x<L$.

$a_0 = L$ and $a_n = \frac{2L}{n^2\pi^2}\{(-1)^n - 1\}$.