SCM 228 TUTORIAL 7

METHOD OF FROBENIUS

If $x = 0$ is a regular singular point of a D.E, there is a solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

LEGENDRE'S EQUATION

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

BESSEL'S EQUATION

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

EXAMPLES

1. Zand C, 5.2 №23 p.255 (first part only)

$$9x^2y'' + 9x^2y' + 2y = 0 \quad (1)$$

Try $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\therefore y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$\therefore y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$\therefore$ (1) becomes

$$\sum_{n=0}^{\infty} \left[9(n+r)(n+r-1)a_n x^{n+r} + 9(n+r)a_n x^{n+r+1} + 2a_n x^{n+r} \right] = 0$$

Smallest power of x is x^r

$$\therefore \quad 9(r(r-1)) + 2 = 0$$

$$\therefore 9r^2 - 9r + 2 = 0$$

$$\therefore (3r-1)(3r-2) = 0$$

$\therefore r = \frac{1}{3}, \frac{2}{3}$ i.e. roots do not differ by an integer.

$\therefore$ 2 independent solutions are

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} a_n x^n,$$

$$y_2 = x^{2/3} \sum_{n=0}^{\infty} b_n x^n.$$

2. Zand C, 5.3 №3 p.264

$$4x^2y'' + 4xy' + (4x^2 - 25)y = 0$$

$$\therefore x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$$

i.e. Bessel's equation with

$$\nu^2 = \frac{25}{4} \qquad \therefore \nu = \pm \frac{5}{2}$$

$$\therefore \quad y = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)$$

3. Zand C p.266. №18 (a)

$$\frac{dy}{dx} = x^2 + y^2 \quad (1)$$

Let $y = -\frac{1}{u}\frac{du}{dx}$

$$\therefore \frac{dy}{dx} = -\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2$$

sub. in (1)

$$\therefore -\frac{1}{u}u'' + \frac{1}{u^2}(u')^2 = x^2 + \frac{1}{u^2}(u')^2$$

$$\therefore -\frac{1}{u}u'' = x^2$$

$$\therefore u'' + x^2 u = 0$$

4. Show that $f(x) = e^x$ and $g(x) = x^2 - x - 1$ are orthogonal on the interval $[1,2]$.

$$\int_1^2 (x^2 - x - 1)e^x \, dx$$

$$= \left[(x^2 - 3x + 2)e^x \right]_1^2$$

(Tables)

$$= \left[(x-1)(x-2)e^x \right]_1^2$$

$$= 0$$

$\therefore$ f and g are orthogonal.

EXERCISE

For example 1, find the recurrence relation for the solution corresponding to $r = \frac{1}{3}$.

When $r = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \left[9\left(n+\tfrac{1}{3}\right)\left(n-\tfrac{2}{3}\right)a_n x^{n+1/3} + 9\left(n+\tfrac{1}{3}\right)a_n x^{n+1+1/3} + 2a_n x^{n+1/3} \right] = 0$$

Since $9\left(n+\frac{1}{3}\right)\left(n-\frac{2}{3}\right) + 2$

$$= 9n^2 - 3n = 3n(3n-1)$$

$$\sum_0^{\infty} \left[3n(3n-1)a_n x^n + 3(3n+1)a_n x^{n+1} \right] = 0$$

co-eff$^{\text{ts}}$ of x^{m+1}:

$$3(m+1)(3m+2)a_{m+1} + 3(3m+1)a_m = 0$$

$$\therefore a_{m+1} = \frac{-(3m+1)}{(m+1)(3m+2)} a_m.$$