1. (a) $y = f(x) = |4x-1|$

(i) Domain : all real x.

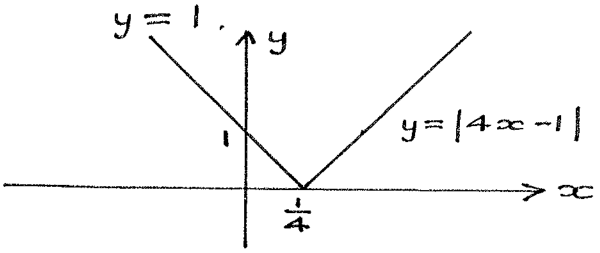
Range : $y \geqslant 0$.

$$y = \begin{cases} 4x-1, & \text{if } 4x-1 \geqslant 0 \\ -(4x-1), & \text{if } 4x-1 < 0 \end{cases}$$

$$\therefore y = \begin{cases} 4x-1, & \text{if } x \geqslant \frac{1}{4} \\ -4x+1, & \text{if } x < \frac{1}{4}. \end{cases}$$

For $y = 4x-1$, when $x = \frac{1}{4}$, $y = 0$.

For $y = -4x+1$, when $x = 0$, $y = 1$.



(ii) Let $f(a) = f(b)$

$\therefore |4a-1| = |4b-1|$

$\therefore 4a-1 = \pm(4b-1)$

i.e. $4a-1 = \begin{cases} 4b-1 \\ -4b+1 \end{cases}$

$\therefore 4a = \begin{cases} 4b \\ -4b+2 \end{cases}$

$\therefore a = b$ or $a = \dfrac{-4b+2}{4} = \dfrac{1-2b}{2}$

Since $a = b$ is not the only solution, $f(x)$ is not one-to-one.

(iii) From the sketch, $f(x)$ is one-to-one for either $x \geqslant \frac{1}{4}$ or $x \leqslant \frac{1}{4}$.

(b) $y = f(x) = \sqrt{10x-25}$, $x \geqslant \frac{5}{2}$.

i.e. $y = \sqrt{10x-25}$; $x \geqslant \frac{5}{2}$, $y \geqslant 0$.

Swapping x and y gives

$x = \sqrt{10y-25}$; $x \geqslant 0$, $y \geqslant \frac{5}{2}$.

$\therefore x^2 = 10y - 25$

$\therefore 10y = x^2 + 25$

$\therefore y = \dfrac{x^2+25}{10}$.

$\therefore f^{-1}(x) = \dfrac{x^2+25}{10}$; $x \geqslant 0$

is the required inverse.

(ii) $y = f^{-1}(x) = \dfrac{x^2+25}{10}$, $x \geqslant 0$

is a parabola (right-half only).

When $x = 0$, $y = \frac{25}{10} = \frac{5}{2}$.

$f(x)$ and $f^{-1}(x)$ intersect when both intersect the line $y = x$, i.e. when

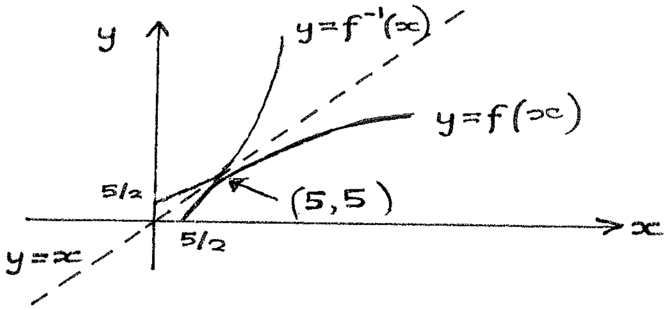
$$\frac{x^2+25}{10} = x$$

$\therefore x^2 + 25 = 10x$

$\therefore x^2 - 10x + 25 = 0$

$\therefore (x-5)^2 = 0$

$\therefore x = 5$.



(c) (i) $\displaystyle\lim_{x\to\infty} \frac{5x^2-2x+1}{7x^2+6}$

$= \displaystyle\lim_{x\to\infty} \frac{x^2(5 - 2/x + 1/x^2)}{x^2(7 + 6/x^2)}$

$= \displaystyle\lim_{x\to\infty} \frac{5 - 2/x + 1/x^2}{7 + 6/x^2}$

$= \dfrac{5-0+0}{7+0} = \dfrac{5}{7}$.

(ii) $\displaystyle\lim_{x\to 4} \frac{2x^2-9x+4}{x^2-16}$

$\left(= \dfrac{32-36+4}{16-16} = \dfrac{0}{0}\right)$

$= \displaystyle\lim_{x\to 4} \frac{(x-4)(2x-1)}{(x-4)(x+4)}$

$= \displaystyle\lim_{x\to 4} \frac{2x-1}{x+4} = \frac{8-1}{4+4}$

$= \dfrac{7}{8}$

(or use L'H.R).

1. (c) (iii)

$$\lim_{x\to 1} \frac{5x^6+2x^3-3x-4}{x^2+x-2}$$

$$\left(= \frac{5+2-3-4}{1+1-2} = \frac{0}{0} \therefore \text{Use L'H.R}\right)$$

$$= \lim_{x\to 1} \frac{30x^5+6x^2-3}{2x+1}$$

$$= \frac{30+6-3}{2+1} = \frac{33}{3} = 11.$$

(iv) $$\lim_{x\to 0} \frac{3e^{4x}-5+2\cos x}{3x}$$

$$\left(= \frac{3-5+2}{0} = \frac{0}{0} \therefore \text{Use L'H.R}\right)$$

$$= \lim_{x\to 0} \frac{12e^{4x}-2\sin x}{3}$$

$$= \frac{12-0}{3} = 4.$$

2. (a) (i) $y = \dfrac{4x-3}{5x-7}$

$$\therefore \frac{dy}{dx} = \frac{4(5x-7)-5(4x-3)}{(5x-7)^2}$$

(Quotient)

$$\therefore \frac{dy}{dx} = \frac{20x-28-20x+15}{(5x-7)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-13}{(5x-7)^2}.$$

(ii) $y = (x\ln x - 4)^5$

Let $u = x\ln x - 4 \quad \therefore y = u^5$

$$\therefore \frac{du}{dx} = x\left(\frac{1}{x}\right) + \ln x$$

(Product)

$$\therefore \frac{du}{dx} = 1+\ln x \; ; \; \frac{dy}{du} = 5u^4$$

$$\therefore \frac{dy}{dx} = 5u^4(1+\ln x) \quad \text{(Chain)}$$

$$\therefore \frac{dy}{dx} = 5(1+\ln x)(x\ln x-4)^4.$$

(iii) $y = (2\sin x - \cos x)e^{2x}$

$$\therefore \frac{dy}{dx} = (2\sin x - \cos x)(2e^{2x}) + (2\cos x + \sin x)e^{2x}$$

(Product)

$$\therefore \frac{dy}{dx} = (4\sin x - 2\cos x)e^{2x} + (2\cos x + \sin x)e^{2x}$$

$$\therefore \frac{dy}{dx} = 5\sin x \, e^{2x}.$$

(iv) $3x^4 - 5xy + \sin y = 4$

$$\therefore \frac{d}{dx}\left[3x^4 - 5xy + \sin y\right] = \frac{d}{dx}(4)$$

$$\therefore 12x^3 - 5\left[x\frac{dy}{dx} + y\right] + \frac{d}{dx}(\sin y) = 0$$

(Product)

$$\therefore 12x^3 - 5x\frac{dy}{dx} - 5y + \cos y\frac{dy}{dx} = 0$$

(Chain)

$$\therefore 12x^3 - 5y = 5x\frac{dy}{dx} - \cos y\frac{dy}{dx}$$

$$\therefore 12x^3 - 5y = (5x - \cos y)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{12x^3-5y}{5x-\cos y}.$$

(b) $y = \dfrac{(2x^3-6x+1)^{1/6}}{(3x^2-7)^{5/2}}$

$$\therefore \ln y = \ln\left[\frac{(2x^3-6x+1)^{1/6}}{(3x^2-7)^{5/2}}\right]$$

$$= \ln(2x^3-6x+1)^{1/6} - \ln(3x^2-7)^{5/2}$$

$$= \frac{1}{6}\ln(2x^3-6x+1) - \frac{5}{2}\ln(3x^2-7)$$

$$\therefore \frac{d}{dx}(\ln y) = \frac{1}{6}\left(\frac{6x^2-6}{2x^3-6x+1}\right) - \frac{5}{2}\left(\frac{6x}{3x^2-7}\right)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{x^2-1}{2x^3-6x+1} - \frac{15x}{3x^2-7}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{x^2-1}{2x^3-6x+1} - \frac{15x}{3x^2-7}\right].$$

(c) (i) $y = \sin^{-1}(\sqrt{x})$

Let $u = \sqrt{x} \quad \therefore y = \sin^{-1} u$

$$\therefore \frac{du}{dx} = \frac{1}{2\sqrt{x}} \; , \; \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-u^2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

2.(c) (ii)

$y = x\cosh x - \sinh x$

$\therefore \dfrac{dy}{dx} = x\sinh x + \cosh x - \cosh x$ (Product)

$\therefore \dfrac{dy}{dx} = x\sinh x.$

(iii) $y = \displaystyle\int_0^x \frac{\sqrt{t^2+3}}{t+e^t}\,dt$

$\therefore \dfrac{dy}{dx} = \dfrac{\sqrt{x^2+3}}{x+e^x}$

(by the Fundamental Theorem of Calculus).

3. (a) (i) By definition:

$\cosh x = \frac{1}{2}(e^x + e^{-x})$;

$\sinh x = \frac{1}{2}(e^x - e^{-x})$.

(ii) To Prove:

$\ln(\cosh x - \sinh x) + x = 0$

Proof: L.S. $= \ln(\cosh x - \sinh x) + x$

$= \ln\left[\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})\right] + x$

$= \ln\left[\frac{1}{2}(e^x + e^{-x} - e^x + e^{-x})\right] + x$

$= \ln\left[\frac{1}{2}(2e^{-x})\right] + x$

$= \ln(e^{-x}) + x$

$= -x + x$

$= 0$

$=$ R.S, as required.

(b) $y = f(x) = x^4 - 8x^2 + 5$ for $-1 \le x \le 3$.

$f(x)$ is continuous and differentiable for all x.

$f'(x) = 4x^3 - 16x$

$= 4x(x^2 - 4)$

$= 4x(x-2)(x+2)$

$\therefore f'(x) = 0$ when $x = -2, 0, 2.$

For $-1 \le x \le 3$, consider $x = -1, 0, 2, 3$.

$f(-1) = 1 - 8 + 5 = -2.$

$f(0) = 5.$

$f(2) = 16 - 32 + 5 = -11.$

$f(3) = 81 - 72 + 5 = 14.$

$\therefore$ Absolute maximum is 14 (when $x = 3$).

Absolute minimum is -11 (when $x = 2$).

(c) $y = f(x) = \dfrac{2(x^2-4)}{(x-1)^2}$.

(i) domain: all x, except where $(x-1)^2 = 0$

i.e. all x, except $x = 1$.

(ii) vertical asymptotes: $x = 1$

(as $x \to 1$, $y \to \frac{2(1-4)}{0} = \frac{-6}{0}$)

(iii) symmetry: $f(x) = \dfrac{2(x^2-4)}{(x-1)^2}$

$\therefore f(-x) = \dfrac{2[(-x)^2-4]}{(-x-1)^2}$

$\therefore f(-x) = \dfrac{2(x^2-4)}{(-x-1)^2} \neq \begin{cases} f(x) \\ -f(x) \end{cases}$

$\therefore f(x)$ is neither even nor odd.

(iv) intercepts: When $x = 0$,

$y = \dfrac{2(0-4)}{(0-1)^2} = -8.$

When $y = 0$, $2(x^2-4) = 0$

$\therefore (x+2)(x-2) = 0$

$\therefore x = \pm 2.$

$\therefore$ Intercepts: $(0,-8)$, $(-2,0)$ and $(2,0)$.

(v) as $x \to \pm\infty$, $y = \dfrac{2(x^2-4)}{(x-1)^2}$

$\therefore y \sim \dfrac{2x^2}{x^2} = 2.$

Curve crosses $y = 2$ when

$\dfrac{2(x^2-4)}{(x-1)^2} = 2$

$\therefore 2(x^2-4) = 2(x-1)^2$

$\therefore x^2 - 4 = (x-1)^2$

$\therefore x^2 - 4 = x^2 - 2x + 1$

3.(c) (v) (continued)

$\therefore -4 = -2x+1$

$\therefore -5 = -2x$

$\therefore x = \frac{5}{2}$.

(vi) sign of y : as $(x-1)^2 \geq 0$, the sign of y is the sign of $2(x^2-4)$

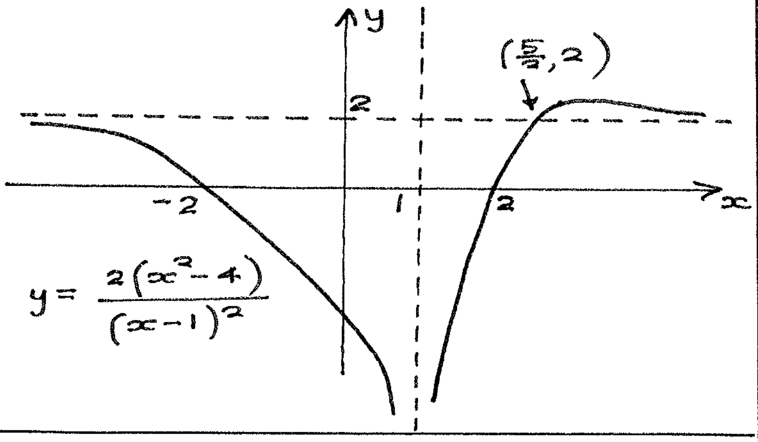
$\therefore y < 0$ when $x^2 - 4 < 0$

$\therefore x^2 < 4$

$\therefore |x| < 2$

$\therefore -2 < x < 2$.

Similarly, $y > 0$ when $x < -2$ or $x > 2$.



4. (a) (i) $I = \int x\sqrt{3x^2+5}\,dx$

Let $u = 3x^2+5$

$\therefore \frac{du}{dx} = 6x$

$\therefore du = 6x\,dx$

$\therefore \frac{1}{6}du = x\,dx$

and $\sqrt{3x^2+5} = \sqrt{u}$

$\therefore I = \frac{1}{6}\int \sqrt{u}\,du$

$= \frac{1}{6}\int u^{1/2}\,du$

$= \frac{1}{6} \frac{u^{3/2}}{3/2} + C$

$= \frac{1}{9}u^{3/2} + C$

$\therefore I = \frac{1}{9}(3x^2+5)^{3/2} + C$.

(ii) $I = \int (3x-5)e^{3x}\,dx$

Let $u = 3x-5$; $\frac{dv}{dx} = e^{3x}$

$\therefore \frac{du}{dx} = 3$; $v = \frac{1}{3}e^{3x}$.

Integrating by parts,

$I = \frac{1}{3}(3x-5)e^{3x} - \frac{1}{3}\int 3e^{3x}\,dx$

$= \frac{(3x-5)e^{3x}}{3} - \int e^{3x}\,dx$

$= \frac{(3x-5)e^{3x}}{3} - \frac{1}{3}e^{3x} + C$

$= \frac{(3x-6)e^{3x}}{3} + C$

$\therefore I = (x-2)e^{3x} + C$

(iii) $I = \int \frac{\cos x}{(4-\sin x)^3}\,dx$

Let $u = 4 - \sin x$

$\therefore \frac{du}{dx} = -\cos x$

$\therefore du = -\cos x\,dx$

$\therefore -du = \cos x\,dx$

and $(4-\sin x)^3 = u^3$

$\therefore I = -\int \frac{1}{u^3}\,du$

$= -\int u^{-3}\,du$

$= -\frac{u^{-2}}{-2} + C$

$= \frac{1}{2u^2} + C$

$\therefore I = \frac{1}{2(4-\sin x)^2} + C$.

(b) $I = \int_0^5 \frac{1}{x^2+25}\,dx$

$= \left[\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)\right]_0^5$

(S.I.3 ; $a=5$)

$= \frac{1}{5}\left(\tan^{-1}1 - \tan^{-1}0\right)$

$= \frac{1}{5}\left(\frac{\pi}{4} - 0\right)$

$\therefore I = \frac{\pi}{20}$.

4.(c) $I = \int \frac{x-7}{x^2-4x+3}\,dx$.

As $x^2-4x+3 = (x-3)(x-1)$

$$\frac{x-7}{x^2-4x+3} = \frac{x-7}{(x-3)(x-1)}$$

$$= \frac{A}{x-3} + \frac{B}{x-1}$$

$$\therefore x-7 = A(x-1)+B(x-3)$$

When $x=3$,

$-4 = 2A \qquad \therefore A=-2$.

When $x=1$,

$-6 = -2B \qquad \therefore B = 3$.

$$\therefore I = \int\left(\frac{3}{x-1} - \frac{2}{x-3}\right)dx$$

$$\therefore I = 3\ln|x-1| - 2\ln|x-3| + C$$

$$\left(= \ln\left|\frac{(x-1)^3}{(x-3)^2}\right| + c\right).$$

5.(a) (i) $I = \int_0^{\pi/2} \frac{3\cos x}{\sqrt{16+9\sin^2 x}}\,dx$

Let $u = 3\sin x$

$\therefore \frac{du}{dx} = 3\cos x$

$\therefore du = 3\cos x\,dx$

and $\sqrt{16+9\sin^2 x} = \sqrt{16+u^2}$

Terminals: $x = 0 \Rightarrow u = 0$.

$x = \frac{\pi}{2} \Rightarrow u = 3$.

$$\therefore I = \int_0^3 \frac{1}{\sqrt{16+u^2}}\,du$$

(ii) $I = \left[\ln\{u + \sqrt{u^2+16}\}\right]_0^3$

$(5.I.6\ ;\ a = 4)$

$$= \ln\{3+\sqrt{9+16}\} - \ln\{0+\sqrt{16}\}$$

$$= \ln(3+5) - \ln 4$$

$$= \ln 8 - \ln 4$$

$$= \ln\left(\frac{8}{4}\right)$$

$$\therefore I = \ln 2.$$

(b) $y = 2x^2 - 8x$

$= 2x(x-4)$

is a parabola.

When $x=0$, $y=0$.

When $y=0$, $x = 0, 4$.

$y = 2x-8 = 2(x-4)$

is a straight line.

When $x=0$, $y=-8$.

When $y=0$, $x=4$.

Intersections occur when

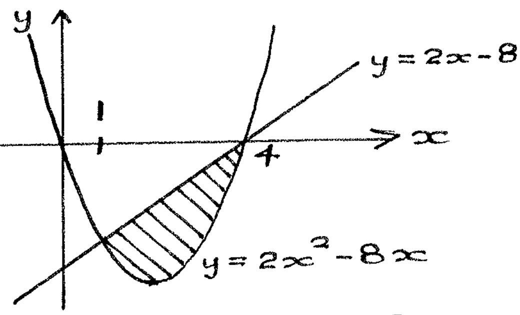
$$2x^2-8x = 2x-8$$

$$\therefore 2x^2-10x+8=0$$

$$\therefore x^2-5x+4=0$$

$$\therefore (x-1)(x-4)=0$$

$$\therefore x = 1, \text{ and } x = 4.$$



For $1 \le x \le 4$, $2x-8 \geqslant 2x^2-8x$

$$\therefore 2x-8-(2x^2-8x)$$

$$= 2x-8-2x^2+8x$$

$$= 10x-8-2x^2$$

$$= 2(5x-4-x^2)$$

$$\therefore \text{Area, } A = 2\int_1^4 (5x-4-x^2)\,dx$$

$$= 2\left[\frac{5x^2}{2} - 4x - \frac{x^3}{3}\right]_1^4$$

$$= 2\left[\left(40-16-\frac{64}{3}\right) - \left(\frac{5}{2}-4-\frac{1}{3}\right)\right]$$

$$= 2\left(24 - \frac{64}{3} - \frac{5}{2} + 4 + \frac{1}{3}\right)$$

$$= 2\left(28 - \frac{63}{3} - \frac{5}{2}\right)$$

$$= 2\left(28-21-\frac{5}{2}\right)$$

$$= 2\left(7-\frac{5}{2}\right)$$

$$= 14-5$$

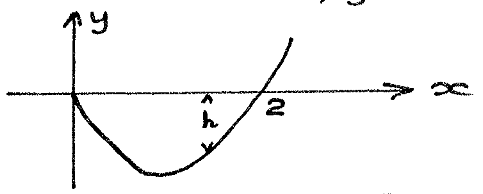
$$\therefore A = 9.$$

5.(c). $y = (x-2)\sqrt{x}$

is defined for $x \geq 0$.

$y = 0$ when $x = 0$, $x = 2$.

For $0 < x < 2$, $y < 0$.



$\therefore h = -y$ and $h^2 = y^2$

$\therefore h^2 = (x-2)^2 x$

$= (x^2 - 4x + 4)x$

$= x^3 - 4x^2 + 4x$.

$\therefore$ Volume, V, is given by

$$V = \pi \int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= \pi \left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2\right]_0^2$$

$$= \pi \left[\left(4 - \frac{32}{3} + 8\right) - 0\right]$$

$$= \pi \left(12 - \frac{32}{3}\right)$$

$$= \pi \left(\frac{36}{3} - \frac{32}{3}\right)$$

$$\therefore V = \frac{4\pi}{3}.$$

6. (a) (i) $a_n = \dfrac{2n + (-1)^n}{2n}$

$\therefore a_1 = \dfrac{2-1}{2} = \dfrac{1}{2}$.

$a_2 = \dfrac{4+1}{4} = \dfrac{5}{4}$.

$a_3 = \dfrac{6-1}{6} = \dfrac{5}{6}$.

$a_4 = \dfrac{8+1}{8} = \dfrac{9}{8}$.

(ii) In general,

$$a_n = \begin{cases} \dfrac{2n-1}{2n}, & \text{for } n \text{ odd} \\ \dfrac{2n+1}{2n}, & \text{for } n \text{ even} \end{cases}$$

$$\therefore a_n = \begin{cases} 1 - \dfrac{1}{2n}, & \text{for } n \text{ odd} \\ 1 + \dfrac{1}{2n}, & \text{for } n \text{ even} \end{cases}$$

As $\lim\limits_{n\to\infty}\left(1 - \dfrac{1}{2n}\right) = \lim\limits_{n\to\infty}\left(1 + \dfrac{1}{2n}\right)$

$= 1$

$\lim\limits_{n\to\infty} a_n = 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{(n+7)\,9^n}$

$\therefore a_n = \dfrac{(x+2)^n}{(n+7)\,9^n}$

$\therefore a_{n+1} = \dfrac{(x+2)^{n+1}}{(n+8)\,9^{n+1}}$.

$\therefore \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}(n+7)(9^n)}{(n+8)(9^{n+1})(x+2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+2)(n+7)}{9(n+8)} \right|$$

$$= \frac{|x+2|}{9} \lim_{n\to\infty} \frac{n+7}{n+8}$$

$$= \frac{|x+2|}{9} \lim_{n\to\infty} \frac{n(1+7/n)}{n(1+8/n)}$$

$$= \frac{|x+2|}{9} \lim_{n\to\infty} \frac{1+7/n}{1+8/n}$$

$$= \frac{|x+2|}{9}.$$

For convergence,

$$\frac{|x+2|}{9} < 1$$

$\therefore |x+2| < 9$

$\therefore -9 < x+2 < 9$

$\therefore -11 < x < 7$.

(c) (i) For $f(x) = (1+x)^{-3/4}$,

$f(x) = (1+x)^{-3/4}$; $f(0) = 1$.

$f'(x) = \dfrac{-3}{4}(1+x)^{-7/4}$; $f'(0) = \dfrac{-3}{4}$.

$f''(x) = \dfrac{21}{16}(1+x)^{-11/4}$; $f''(0) = \dfrac{21}{16}$.

$\therefore f(x) = f(0) + f'(0)x + \dfrac{1}{2!}f''(0)x^2 + \ldots$

$= 1 - \dfrac{3}{4}x + \dfrac{1}{2}\left(\dfrac{21}{16}\right)x^2 + \ldots$

$\therefore (1+x)^{-3/4} = 1 - \dfrac{3x}{4} + \dfrac{21x^2}{32} + \ldots$

(ii) When $x = 0.08$,

$(1.08)^{-3/4} \simeq 1 - \dfrac{3(0.08)}{4} + \dfrac{21(0.08)^2}{32}$

$\simeq 1 - 0.06 + 0.0042 \simeq 0.9442$.

6. (d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

for $-\infty < x < \infty$.

(i) $\frac{x - \sin x}{x^3} = \frac{1}{x^3}\left[x - \sin x\right]$

$= \frac{1}{x^3}\left[x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)\right]$

$= \frac{1}{x^3}\left[x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right]$

$= \frac{1}{x^3}\left[\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right]$

$= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \ldots$

(for $-\infty < x < \infty$)

$\therefore \frac{x - \sin x}{x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+3)!}$

for $x \neq 0$.

(ii) $\frac{x - \sin x}{x^3} \simeq \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!}$

$= \frac{1}{6} - \frac{x^2}{120} + \frac{x^4}{5040}$.

$\therefore I = \int_0^1 \frac{x - \sin x}{x^3}\, dx$

$\simeq \int_0^1 \left(\frac{1}{6} - \frac{x^2}{120} + \frac{x^4}{5040}\right) dx$

$= \left[\frac{x}{6} - \frac{x^3}{360} + \frac{x^5}{25{,}200}\right]_0^1$

$= \left(\frac{1}{6} - \frac{1}{360} + \frac{1}{25{,}200}\right) - 0$

$\therefore I \simeq 0.1639$.

7. (a)(i) $\frac{dy}{dx} = \frac{4y^{3/4}}{x^{1/2}}$;

with $y(4) = 1$.

$\therefore dy = \frac{4y^{3/4}}{x^{1/2}}\, dx$

$\therefore \frac{1}{y^{3/4}}\, dy = \frac{4}{x^{1/2}}\, dx$

$\therefore \int y^{-3/4}\, dy = \int 4x^{-1/2}\, dx$

$\therefore 4y^{1/4} = 8x^{1/2} + C$

$\therefore y^{1/4} = 2x^{1/2} + K$

(where $K = \frac{C}{4}$).

When $x = 4$, $y = 1$

$\therefore 1^{1/4} = 2 \times 4^{1/2} + K$

$\therefore 1 = 4 + K \quad \therefore K = -3$

$\therefore y^{1/4} = 2x^{1/2} - 3$.

$\therefore$ Sol$^{\underline{n}}$: $y = (2x^{1/2} - 3)^4$,

i.e. $y = (2\sqrt{x} - 3)^4$.

(ii) $\frac{dy}{dx} + \frac{3y}{x} = \frac{4x+9}{x}$; (1)

with $y(1) = 2$.

(1) is linear with $p(x) = \frac{3}{x}$

$\therefore$ Integrating factor $\mu(x)$ is given by $\mu = e^{\int \frac{3}{x} dx}$

$\therefore \mu = e^{3\ln|x| + C}$.

Choosing $C = 0$, and $x > 0$,

$\mu = e^{3\ln x} = e^{\ln(x^3)}$

$\therefore \mu = x^3$.

(1) by x^3 gives

$x^3 \frac{dy}{dx} + 3x^2 y = x^2(4x + 9)$

$\therefore \frac{d}{dx}\left[x^3 y\right] = 4x^3 + 9x^2$

$\therefore x^3 y = \int (4x^3 + 9x^2)\, dx$

$\therefore x^3 y = x^4 + 3x^3 + K$

$\therefore y = \frac{x^4 + 3x^3 + K}{x^3}$.

When $x = 1$, $y = 2$

$\therefore 2 = 1 + 3 + K \quad \therefore K = -2$.

$\therefore$ Sol$^{\underline{n}}$: $y = \frac{x^4 + 3x^3 - 2}{x^3}$,

i.e. $y = x + 3 - \frac{2}{x^3}$.

7. (b) $\underset{\sim}{A} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$, and

$\underset{\sim}{B} = 8\underset{\sim}{i} - 4\underset{\sim}{j} - \underset{\sim}{k}$.

(i) $|\underset{\sim}{A}| = \sqrt{2^2 + (-4)^2 + 4^2}$

$= \sqrt{4 + 16 + 16} = \sqrt{36}$

$\therefore |\underset{\sim}{A}| = 6$.

$|\underset{\sim}{B}| = \sqrt{8^2 + (-4)^2 + (-1)^2}$

$= \sqrt{64 + 16 + 1} = \sqrt{81}$

$\therefore |\underset{\sim}{B}| = 9$.

(ii) $\underset{\sim}{A} \cdot \underset{\sim}{B} = (2)(8) + (-4)(-4) + (4)(-1)$

$= 16 + 16 - 4$

$\therefore \underset{\sim}{A} \cdot \underset{\sim}{B} = 28$.

(iii) $\cos\theta = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|} = \dfrac{28}{(6)(9)}$

$= \dfrac{28}{54} = \dfrac{14}{27}$.

(iv) $\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -4 & 4 \\ 8 & -4 & -1 \end{vmatrix}$

$= \underset{\sim}{i} \begin{vmatrix} -4 & 4 \\ -4 & -1 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 2 & 4 \\ 8 & -1 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 2 & -4 \\ 8 & -4 \end{vmatrix}$

$= (4 + 16)\underset{\sim}{i} - (-2 - 32)\underset{\sim}{j} + (-8 + 32)\underset{\sim}{k}$

$\therefore \underset{\sim}{A} \times \underset{\sim}{B} = 20\underset{\sim}{i} + 34\underset{\sim}{j} + 24\underset{\sim}{k}$.

8. (a)(i) Given $P(4, -3, 2)$, $Q(6, 0, 4)$ and $R(6, -6, 2)$, two vectors in the plane are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$\overrightarrow{PQ} = (6 - 4)\underset{\sim}{i} + (0 + 3)\underset{\sim}{j} + (4 - 2)\underset{\sim}{k}$

$\therefore \overrightarrow{PQ} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$.

$\overrightarrow{PR} = (6 - 4)\underset{\sim}{i} + (-6 + 3)\underset{\sim}{j} + (2 - 2)\underset{\sim}{k}$

$\therefore \overrightarrow{PR} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

A vector perpendicular to the plane is $\underset{\sim}{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$\therefore \underset{\sim}{N} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & 2 \\ 2 & -3 & 0 \end{vmatrix}$

$= \underset{\sim}{i} \begin{vmatrix} 3 & 2 \\ -3 & 0 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix}$

$= (0 + 6)\underset{\sim}{i} - (0 - 4)\underset{\sim}{j} + (-6 - 6)\underset{\sim}{k}$

$\therefore \underset{\sim}{N} = 6\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k}$

is a vector perpendicular to the plane.

(ii) Area of triangle PQR is

$\frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}|$

$= \frac{1}{2} |6\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k}|$

$= |3\underset{\sim}{i} + 2\underset{\sim}{j} - 6\underset{\sim}{k}|$

$= \sqrt{3^2 + 2^2 + (-6)^2}$

$= \sqrt{9 + 4 + 36} = \sqrt{49}$

$\therefore$ Area is 7.

(iii) As $\underset{\sim}{N} = 6\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k}$ is a normal to the plane, the equation is

$6x + 4y - 12z = D$

As $Q(6, 0, 4)$ is in the plane,

$6 \times 6 + 4 \times 0 - 12 \times 4 = D$

$\therefore D = 36 - 48 = -12$.

$\therefore$ The equation is

$6x + 4y - 12z = -12$

i.e. $3x + 2y - 6z = -6$.

(b) Given $P(9, -1, -4)$ and $Q(5, 0, 3)$, $\underset{\sim}{v} = \overrightarrow{PQ}$ is parallel to the line

$\therefore \underset{\sim}{v} = \overrightarrow{PQ} = (5 - 9)\underset{\sim}{i} + (0 + 1)\underset{\sim}{j} + (3 + 4)\underset{\sim}{k}$

$\therefore \underset{\sim}{v} = -4\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}$

Since $P(9, -1, -4)$ is on the line, the parametric equations are:

$x = 9 - 4t$; $y = -1 + t$; $z = -4 + 7t$

(or equivalent).

8.(c) Substituting the equations of the line:

$x=-6+5t$; $y=-4+3t$; $z=1-2t$

into the equation of the plane

$3x-4y-z=7$ gives

$3(-6+5t)-4(-4+3t)-(1-2t)=7$

$\therefore -18+15t+16-12t-1+2t=7$

$\therefore 5t-3=7$

$\therefore 5t=10$

$\therefore t=2$

When $t=2$,

$x=-6+10=4$

$y=-4+6=2$

$z=1-4=-3$.

$\therefore$ The point of intersection is $(4,2,-3)$.

(d) Given $\underset{\sim}{A}=2\underset{\sim}{i}+3\underset{\sim}{j}+\underset{\sim}{k}$, $\underset{\sim}{B}=4\underset{\sim}{i}-2\underset{\sim}{j}$, and $\underset{\sim}{C}=\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$,

$V=|\underset{\sim}{A}\cdot(\underset{\sim}{B}\times\underset{\sim}{C})|$

where $\underset{\sim}{A}\cdot(\underset{\sim}{B}\times\underset{\sim}{C})=\begin{vmatrix}2 & 3 & 1\\ 4 & -2 & 0\\ 1 & 2 & 2\end{vmatrix}$

$=2\begin{vmatrix}-2 & 0\\ 2 & 2\end{vmatrix}-3\begin{vmatrix}4 & 0\\ 1 & 2\end{vmatrix}+1\begin{vmatrix}4 & -2\\ 1 & 2\end{vmatrix}$

$=2(-4-0)-3(8-0)+1(8+2)$

$=-8-24+10$

$=-22$.

$\therefore V=|-22|$

$=22$.