1. (a) $f(x) = |2x-1|$

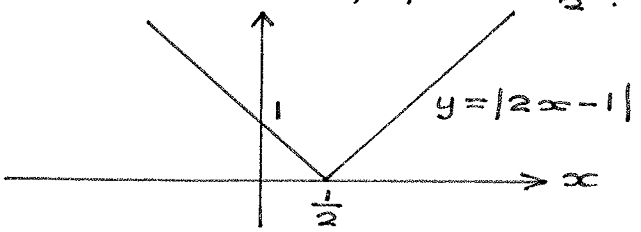
(i) Domain : all real x

Range : $y \geqslant 0$.

By the definition of modulus,

$$y = \begin{cases} 2x-1, & \text{if } 2x-1 \geqslant 0 \\ -(2x-1), & \text{if } 2x-1 < 0 \end{cases}$$

$$\therefore y = \begin{cases} 2x-1, & \text{if } x \geqslant \frac{1}{2} \\ -2x+1, & \text{if } x < \frac{1}{2}. \end{cases}$$



(ii) Let $f(a) = f(b)$

$\therefore |2a-1| = |2b-1|$

$\therefore 2a-1 = \pm(2b-1)$

$\therefore 2a-1 = 2b-1$

or $2a-1 = -2b+1$

$\therefore$ $2a = 2b$ $\quad \therefore a = b$, or

$2a = -2b+2 \quad \therefore a = -b+1$.

Since $a=b$ is not the only solution, $f(x)$ is not one-to-one.

(iii) From the sketch, $f(x)$ can be made one-to-one by restricting the domain to either $x \geqslant \frac{1}{2}$ or $x \leqslant \frac{1}{2}$.

(b) $y = f(x) = \sqrt{6x-8}$;

$x \geqslant \frac{4}{3}$, $y \geqslant 0$.

(i) Swapping x and y gives

$x = \sqrt{6y-8}$; $y \geqslant \frac{4}{3}$, $x \geqslant 0$

$\therefore x^2 = 6y-8$

$\therefore 6y = x^2+8$

$\therefore y = f^{-1}(x) = \dfrac{x^2+8}{6}$; $x \geqslant 0$

is the inverse function.

(ii) $f(x)$ and $f^{-1}(x)$ intersect when both cross the line $y=x$, i.e. when

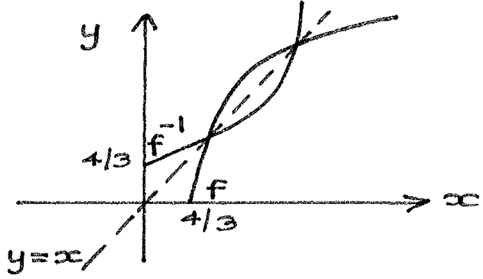
$$\frac{x^2+8}{6} = x$$

$\therefore x^2+8 = 6x$

$\therefore x^2-6x+8 = 0$

$\therefore (x-2)(x-4) = 0$

$\therefore x = 2$ and $x = 4$.



(c) (i) $\lim\limits_{x\to\infty} \dfrac{3x^2-x+2}{4x^2+5}$

$= \lim\limits_{x\to\infty} \dfrac{x^2(3-1/x+2/x^2)}{x^2(4+5/x^2)}$

$= \lim\limits_{x\to\infty} \dfrac{3-1/x+2/x^2}{4+5/x^2}$

$= \dfrac{3-0+0}{4+0} = \dfrac{3}{4}$.

(ii) $\lim\limits_{x\to4} \dfrac{x^2-x-12}{x^2-4x}$ $\left(=\frac{0}{0}\right)$

$= \lim\limits_{x\to4} \dfrac{(x-4)(x+3)}{x(x-4)}$

$= \lim\limits_{x\to4} \dfrac{x+3}{x}$

$= \dfrac{4+3}{4} = \dfrac{7}{4}$.

(iii) $\lim\limits_{x\to1} \dfrac{3x^5+2x^3-x-4}{x^2-1}$

$\left(=\frac{0}{0} \quad \therefore \text{Use L'H.R.}\right)$

$= \lim\limits_{x\to1} \dfrac{15x^4+6x^2-1}{2x}$

$= \dfrac{15+6-1}{2} = \dfrac{20}{2} = 10$.

(iv) $\lim\limits_{x\to0} \dfrac{e^{2x}-3+2\cos x}{5x}$ $\left(=\frac{0}{0} \text{ Use L'H.R}\right)$

$= \lim\limits_{x\to0} \dfrac{2e^{2x}-2\sin x}{5} = \dfrac{2-0}{5}$

$= 2/5$.

2. (a) (i) $y = \dfrac{2x+3}{3x-4}$

$\therefore \dfrac{dy}{dx} = \dfrac{2(3x-4)-3(2x+3)}{(3x-4)^2}$

(Quotient Rule)

$\therefore \dfrac{dy}{dx} = \dfrac{6x-8-6x-9}{(3x-4)^2}$

$= \dfrac{-17}{(3x-4)^2}.$

(ii) $y = (x\sin x + \cos x)^4$

Let $u = x\sin x + \cos x$

$\therefore y = u^4$

$\therefore \dfrac{du}{dx} = x\cos x + \sin x$
$- \sin x$ (Product)

$\therefore \dfrac{du}{dx} = x\cos x$

$\dfrac{dy}{du} = 4u^3$

$\therefore \dfrac{dy}{dx} = 4u^3(x\cos x)$

(Chain Rule)

$\therefore \dfrac{dy}{dx} = 4(x\cos x\,(x\sin x+\cos x)^3$

(iii) $y = (9x^2-6x+2)e^{3x}$

$\therefore \dfrac{dy}{dx} = 3(9x^2-6x+2)e^{3x}$
$+ (18x-6)e^{3x}$

(Product Rule)

$\therefore \dfrac{dy}{dx} = (27x^2-18x+6+18x-6)e^{3x}$

$\therefore \dfrac{dy}{dx} = 27x^2e^{3x}.$

(iv) $5x^3 - 7xy - y^2 = 1.$

$\therefore \dfrac{d}{dx}\left[5x^3-7xy-y^2\right] = \dfrac{d}{dx}(0)$

$\therefore 15x^2 - 7\left[x\dfrac{dy}{dx}+y\right] - \dfrac{d}{dx}(y^2) = 0$

(Product Rule)

$\therefore 15x^2 - 7x\dfrac{dy}{dx} - 7y - 2y\dfrac{dy}{dx} = 0$

(Chain Rule)

$\therefore 15x^2 - 7y = 7x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx}$

$\therefore (7x+2y)\dfrac{dy}{dx} = 15x^2 - 7y$

$\therefore \dfrac{dy}{dx} = \dfrac{15x^2-7y}{7x+2y}.$

(b) $y = \dfrac{(5x-2)^{1/5}}{(x^3-3x+4)^{2/3}}$

$\therefore \ln y = \ln\left[\dfrac{(5x-2)^{1/5}}{(x^3-3x+4)^{2/3}}\right]$

$= \ln\left[(5x-2)^{1/5}\right] - \ln\left[(x^3-3x+4)^{2/3}\right]$

$\therefore \ln y = \frac{1}{5}\ln(5x-2) - \frac{2}{3}\ln(x^3-3x+4)$

$\therefore \dfrac{d}{dx}(\ln y) = \frac{1}{5}\left(\dfrac{5}{5x-2}\right) - \frac{2}{3}\left(\dfrac{3x^2-3}{x^3-3x+4}\right)$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{5x-2} - \dfrac{2(x^2-1)}{x^3-3x+4}.$

$\therefore \dfrac{dy}{dx} = y\left\{\dfrac{1}{5x-2} - \dfrac{2(x^2-1)}{x^3-3x+4}\right\}.$

(c) (i) $y = \sin^{-1}(x^2)$

Let $u = x^2$ $\therefore y = \sin^{-1}u$

$\therefore \dfrac{du}{dx} = 2x$; $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1-u^2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{2x}{\sqrt{1-u^2}}$

(Chain Rule)

$\therefore \dfrac{dy}{dx} = \dfrac{2x}{\sqrt{1-x^4}}.$

(ii) $y = x^3\cosh x$

$\therefore \dfrac{dy}{dx} = x^3\sinh x + 3x^2\cosh x$

(Product Rule).

(iii) $y = \displaystyle\int_0^x \dfrac{\ln(t+1)}{3+\cos t}\,dt$

$\therefore \dfrac{dy}{dx} = \dfrac{\ln(x+1)}{3+\cos x}.$

3. (a) (i) $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x}).$

(ii) To Prove : $\ln(\cosh x + \sinh x) = x.$

Proof : L.S. $= \ln(\cosh x + \sinh x)$

$= \ln\left[\frac{1}{2}(e^x+e^{-x}) + \frac{1}{2}(e^x-e^{-x})\right]$

$= \ln\left[\frac{1}{2}(e^x+e^{-x}+e^x-e^{-x})\right]$

$= \ln\left[\frac{1}{2}(2e^x)\right]$

$= \ln(e^x)$

$= x$

= R.S, as required.

3.(b) $f(x) = x^3 - 12x + 1$

for $-3 \leq x \leq 5$.

$f'(x) = 3x^2 - 12$

$= 3(x^2 - 4)$

$= 3(x-2)(x+2)$

$\therefore f'(x) = 0$ when $x = \pm 2$.

$\therefore$ Consider $x = -3, -2, 2, 5$.

$f(-3) = -27 + 36 + 1 = 10$.

$f(-2) = -8 + 24 + 1 = 17$.

$f(2) = 8 - 24 + 1 = -15$.

$f(5) = 125 - 60 + 1 = 66$.

$\therefore$ Absolute maximum is 66 (when $x = 5$).

Absolute minimum is -15 (when $x = 2$).

(c) $y = f(x) = \dfrac{4(x^2-9)}{(x-6)^2}$.

(i) Domain : All x except where $(x-6)^2 = 0$

i.e. all x except $x = 6$.

(ii) Vertical asymptote : $x = 6$

(as $x \to 6$, $y \to \frac{4 \times 27}{0}$)

(iii) Symmetry : $f(x) = \dfrac{4(x^2-9)}{(x-6)^2}$

$\therefore f(-x) = \dfrac{4[(-x)^2 - 9]}{(-x-6)^2}$

$= \dfrac{4(x^2-9)}{(-x-6)^2}$

$\therefore f(-x) \neq \begin{cases} f(x) \\ -f(x) \end{cases}$.

$\therefore f(x)$ is neither even nor odd.

(iv) Intercepts : When $x = 0$,

$y = \dfrac{4(0-9)}{(0-6)^2} = \dfrac{-36}{36} = -1$.

When $y = 0$, $4(x^2-9) = 0$

$\therefore (x+3)(x-3) = 0$

$\therefore x = \pm 3$.

$\therefore$ Intercepts are $(0,-1)$, $(-3,0)$ and $(3,0)$.

(v) As $x \to \pm\infty$,

$y = \dfrac{4(x^2-9)}{(x-6)^2} \sim \dfrac{4x^2}{x^2} = 4$.

$y = f(x)$ crosses $y = 4$ when

$\dfrac{4(x^2-9)}{(x-6)^2} = 4$

$\therefore 4(x^2-9) = 4(x-6)^2$

$\therefore x^2 - 9 = (x-6)^2$

$\therefore x^2 - 9 = x^2 - 12x + 36$

$\therefore -9 = -12x + 36$

$\therefore 12x = 45$

$\therefore x = \dfrac{45}{12} = \dfrac{15}{4}$.

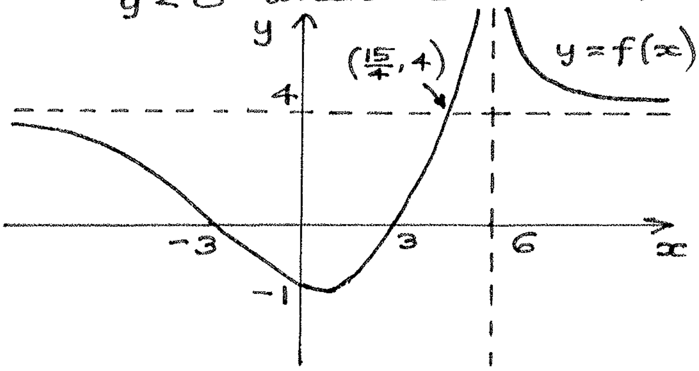
(vi) Sign of y is sign of $x^2 - 9$ (as $(x-6)^2 \geq 0$)

$\therefore y > 0$ when $x^2 - 9 > 0$

i.e. $x^2 > 9$

i.e. $x > 3$ and $x < -3$.

Similarly, $y < 0$ when $-3 < x < 3$.



4. (a) (i) $I = \displaystyle\int \frac{2x}{3x^2-7}\,dx$

Let $u = 3x^2 - 7$

$\therefore \dfrac{du}{dx} = 6x \quad \therefore du = 6x\,dx$

$\therefore \frac{1}{3}du = 2x\,dx$

$\therefore I = \dfrac{1}{3}\displaystyle\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u| + C$

$\therefore I = \frac{1}{3}\ln|3x^2 - 7| + C$.

4.(a) (ii) $I = \int (2x+1)e^{2x}dx$

Let $u = 2x+1$; $\frac{dv}{dx} = e^{2x}$

$\therefore \frac{du}{dx} = 2$; $v = \frac{1}{2}e^{2x}$,

and integrate by parts.

$\therefore I = \frac{1}{2}(2x+1)e^{2x}$

$- \frac{1}{2}\int 2e^{2x}dx$

$= \frac{2x+1}{2}e^{2x} - \int e^{2x}dx$

$= \frac{2x+1}{2}e^{2x} - \frac{1}{2}e^{2x} + C$

$= \frac{2x+1-1}{2}e^{2x} + C$

$\therefore I = xe^{2x} + C$.

(iii) $I = \int \frac{\cos x}{\sqrt{4+\sin x}}\,dx$

Let $u = 4+\sin x$

$\therefore \frac{du}{dx} = \cos x$

$\therefore du = \cos x\,dx$

$\therefore I = \int \frac{1}{\sqrt{u}}\,du$

$= \int u^{-1/2}du$

$= 2u^{1/2} + C$

$\therefore I = 2\sqrt{4+\sin x} + C$.

(b) $I = \int_0^2 \frac{1}{\sqrt{16-x^2}}\,dx$

$= \left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^2$

(S.I. 4 ; $a = 4$)

$\therefore I = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}0$

$= \frac{\pi}{6} - 0$

$\therefore I = \frac{\pi}{6}$.

(c) $I = \int \frac{x+2}{x^2-5x+4}\,dx$

As $x^2-5x+4 = (x-4)(x-1)$

$\frac{x+2}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$

$\therefore x+2 = A(x-1) + B(x-4)$

When $x = 4$,

$6 = 3A \quad \therefore A = 2$.

When $x = 1$,

$3 = -3B \quad \therefore B = -1$.

$\therefore \frac{x+2}{(x-4)(x-1)} = \frac{2}{x-4} - \frac{1}{x-1}$

$\therefore I = \int \left(\frac{2}{x-4} - \frac{1}{x-1}\right)dx$

$= 2\ln|x-4| - \ln|x-1| + C$

$\therefore I = \ln\left|\frac{(x-4)^2}{x-1}\right| + C$.

5.(a) (i) $I = \int_0^2 \frac{2x}{\sqrt{x^4+9}}\,dx$

Let $u = x^2$

$\therefore \frac{du}{dx} = 2x$

$\therefore du = 2x\,dx$

and $x^4 + 9 = u^2 + 9$

Terminals : $x = 0 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 4$.

$\therefore I = \int_0^4 \frac{1}{\sqrt{u^2+9}}\,du$

(ii) From S.I. 6 ; $a = 3$,

$I = \left[\ln\{u + \sqrt{u^2+9}\}\right]_0^4$

$= \ln(4+\sqrt{16+9}) - \ln(0+\sqrt{9})$

$= \ln(4+5) - \ln 3$

$= \ln 9 - \ln 3$

$= \ln\left(\frac{9}{3}\right)$

$\therefore I = \ln 3$.

(b) $y = x^2 - 2x + 1 = (x-1)^2$

is a parabola.

When $x = 0$, $y = 1$.

When $y = 0$, $x = 1$.

$y = x + 1$

is a straight line.

When $x = 0$, $y = 1$.

When $y = 0$, $x = -1$.

5.(b) (continued)

Intersections occur when

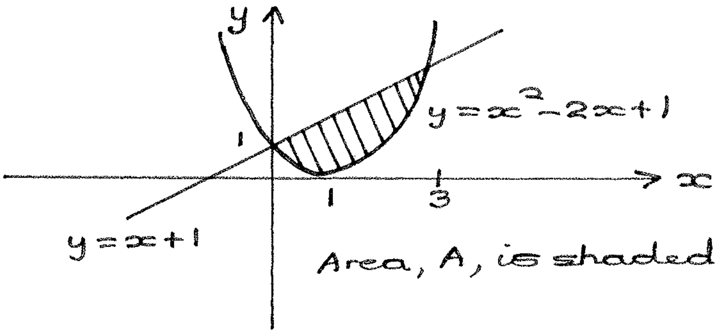
$$x^2-2x+1 = x+1$$

$$\therefore x^2-2x = x$$

$$\therefore x^2-3x = 0$$

$$\therefore x(x-3)=0$$

$$\therefore x=0 \text{ and } x=3.$$



Area, A, is shaded.

$$f(x)-g(x) = x+1-(x^2-2x+1)$$

$$= x+1-x^2+2x-1$$

$$= 3x-x^2.$$

$$\therefore A = \int_0^3 (3x-x^2)\,dx$$

$$= \left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$= \left(\frac{3\times 9}{2}-\frac{27}{3}\right)-0$$

$$= \frac{27}{2}-9$$

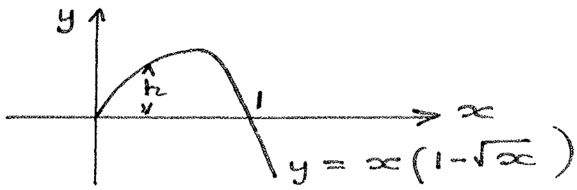
$$= \frac{27}{2}-\frac{18}{2}$$

$$\therefore \text{Area is } \frac{9}{2}.$$

(c) $y = x(1-\sqrt{x})$ is defined for $x \geqslant 0$ only.

$y=0$ when $x=0$ and when $1-\sqrt{x}=0 \quad \therefore x=1$.

For $0<x<1$, $y>0$.



$$h = y = x(1-\sqrt{x})$$

$$\therefore h^2 = x^2(1-\sqrt{x})^2$$

$$= x^2(1-2\sqrt{x}+x)$$

$$\therefore h^2 = x^2-2x^{5/2}+x^3.$$

$\therefore$ The volume, V, is given by

$$V = \pi\int_0^1 h^2\,dx$$

$$\therefore V = \pi\int_0^1 (x^2-2x^{5/2}+x^3)\,dx$$

$$= \pi\left[\frac{x^3}{3}-\frac{4x^{7/2}}{7}+\frac{x^4}{4}\right]_0^1$$

$$= \pi\left[\left(\frac{1}{3}-\frac{4}{7}+\frac{1}{4}\right)-0\right]$$

$$= \pi\left(\frac{28-48+21}{84}\right)$$

$$\therefore \text{Volume is } \frac{\pi}{84}.$$

6.(a) (i) $a_n = \dfrac{3+(-1)^n}{3n}$.

$$a_1 = \frac{3-1}{3} = \frac{2}{3}.$$

$$a_2 = \frac{3+1}{6} = \frac{4}{6} = \frac{2}{3}.$$

$$a_3 = \frac{3-1}{9} = \frac{2}{9}.$$

$$a_4 = \frac{3+1}{12} = \frac{4}{12} = \frac{1}{3}.$$

(ii) In general,

$$a_n = \begin{cases} \dfrac{3-1}{3n} = \dfrac{2}{3n}, & \text{for } n \text{ odd} \\[2mm] \dfrac{3+1}{3n} = \dfrac{4}{3n}, & \text{for } n \text{ even.} \end{cases}$$

Since $\displaystyle\lim_{n\to\infty} \frac{2}{3n} = 0$, and $\displaystyle\lim_{n\to\infty} \frac{4}{3n} = 0$,

$$\lim_{n\to\infty} a_n = 0.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^n}{(n+6)3^n}$

$$\therefore a_n = \frac{(x-4)^n}{(n+6)3^n}; \quad a_{n+1} = \frac{(x-4)^{n+1}}{(n+7)3^{n+1}}$$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}(n+6)(3^n)}{(n+7)(3^{n+1})(x-4)^n}\right|$$

$$= \frac{|x-4|}{3}\lim_{n\to\infty}\frac{n+6}{n+7} = \frac{|x-4|}{3}.$$

For convergence,

$$\frac{|x-4|}{3} < 1$$

$$\therefore |x-4| < 3$$

$$\therefore -3 < x-4 < 3$$

$$\therefore 1 < x < 7$$

is the interval of convergence.

6.(c)(i) $f(x) = (1+x)^{-3/2}$

$\therefore f'(x) = -\frac{3}{2}(1+x)^{-5/2}$

$f''(x) = \frac{15}{4}(1+x)^{-7/2}.$

$\therefore f(0) = 1,$

$f'(0) = -\frac{3}{2},$

$f''(0) = \frac{15}{4}.$

$\therefore f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \ldots$

$= 1 - \frac{3}{2}x + \left(\frac{1}{2}\right)\left(\frac{15}{4}\right)x^2 + \ldots$

$\therefore (1+x)^{-3/2} = 1 - \frac{3x}{2} + \frac{15x^2}{8} + \ldots$

(ii) $1+x = 1.04$ when

$x = 0.04.$

$\therefore (1.04)^{-3/2} \simeq 1 - \frac{3(0.04)}{2} + \frac{15(0.04)^2}{8}$

$= 1 - \frac{0.12}{2} + \frac{15 \times 0.0016}{8}$

$= 1 - 0.06 + 0.003$

$\therefore (1.04)^{-3/2} \simeq 0.943.$

(d) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

for $-\infty < x < \infty$.

(i) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

$\therefore \frac{e^{-x}-1}{x} = \frac{1}{x}\left[e^{-x} - 1\right]$

$= \frac{1}{x}\left[\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right) - 1\right]$

$= \frac{1}{x}\left[-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right]$

$\therefore \frac{e^{-x}-1}{x} = -1 + \frac{x}{2!} - \frac{x^2}{3!} + \ldots$

for $-\infty < x < \infty$.

(ii) $I = \int_0^1 \frac{e^{-x}-1}{x}\,dx$

$\simeq \int_0^1 \left(-1 + \frac{x}{2!} - \frac{x^2}{3!}\right)dx$

$= \int_0^1 \left(-1 + \frac{x}{2} - \frac{x^2}{6}\right)dx$

$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18}\right]_0^1$

$= \left(-1 + \frac{1}{4} - \frac{1}{18}\right) - 0$

$\therefore I \simeq \frac{-36+9-2}{36} = \frac{-29}{36}$

$\therefore I \simeq -0.8056.$

7.(a) $z = 5-4i,\ w = 2+i$

$\therefore \bar{z} = 5+4i.$

(i) $\bar{z} - 2w = 5+4i - 2(2+i)$

$= 5+4i-4-2i$

$= 1+2i.$

(ii) $zw = (5-4i)(2+i)$

$= 5(2+i) - 4i(2+i)$

$= 10+5i-8i-4i^2$

$= 10+5i-8i+4$

$= 14-3i.$

(iii) $\frac{z}{w} = \frac{5-4i}{2+i}$

$= \frac{(5-4i)(2-i)}{(2+i)(2-i)}$

$= \frac{5(2-i) - 4i(2-i)}{2^2+1^2}$

$= \frac{10-5i-8i+4i^2}{4+1}$

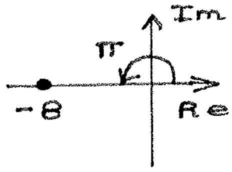
$= \frac{10-5i-8i-4}{5}$

$= \frac{6-13i}{5}.$

(b) To solve $z^3 = -8$,

let $z = r \operatorname{cis}\theta$

$\therefore z^3 = r^3 \operatorname{cis} 3\theta$



From diagram,

$|-8| = 8$

and $\phi = \pi + 2k\pi$.

$\therefore r^3 \operatorname{cis} 3\theta = 8 \operatorname{cis}(\pi + 2k\pi)$

$\therefore r^3 = 8$ and $3\theta = \pi + 2k\pi$

$\therefore r = 2$ and $\theta = \frac{\pi + 2k\pi}{3}$

Choosing $k = 0, 1, 2$ gives

$\theta = \frac{\pi}{3},\ \theta = \frac{\pi + 2\pi}{3} = \pi,$

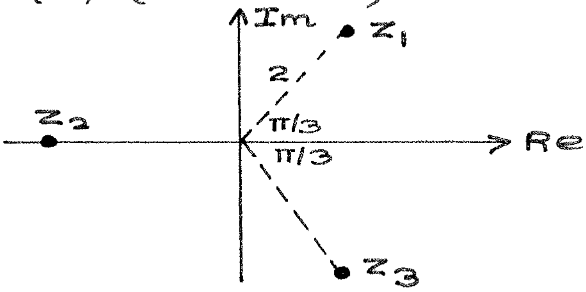
and $\theta = \frac{\pi + 4\pi}{3} = \frac{5\pi}{3}.$

$\therefore$ The solutions are :

$z_1 = 2\operatorname{cis}\frac{\pi}{3};\ z_2 = 2\operatorname{cis}\pi;$

$z_3 = 2\operatorname{cis}\frac{5\pi}{3}.$

7.(b) (continued)



From standard triangles,

$z_1 = 1 + i\sqrt{3}$.

$z_2 = -2$.

From symmetry,

$z_3 = 1 - i\sqrt{3}$.

$\therefore$ The solutions are:

$z_1 = 1 + i\sqrt{3}$, $z_2 = -2i$,

$z_3 = 1 - i\sqrt{3}$.

(c) $\underset{\sim}{A} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$, and

$\underset{\sim}{B} = 4\underset{\sim}{i} + 8\underset{\sim}{j} - \underset{\sim}{k}$.

(i) $|\underset{\sim}{A}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$

$= \sqrt{4+1+4} = \sqrt{9}$

$\therefore |\underset{\sim}{A}| = 3$.

$|\underset{\sim}{B}| = \sqrt{4^2 + 8^2 + (-1)^2}$

$= \sqrt{16+64+1} = \sqrt{81}$

$\therefore |\underset{\sim}{B}| = 9$.

(ii) $\underset{\sim}{A} \cdot \underset{\sim}{B} = (2)(4) + (-1)(8) + (-2)(-1)$

$= 8 - 8 + 2$.

$\therefore \underset{\sim}{A} \cdot \underset{\sim}{B} = 2$.

(iii) $\cos\theta = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|}$

$= \dfrac{2}{3 \times 9}$

$\therefore \cos\theta = \dfrac{2}{27}$.

(iv) $\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -1 & -2 \\ 4 & 8 & -1 \end{vmatrix}$

$= \underset{\sim}{i}\begin{vmatrix} -1 & -2 \\ 8 & -1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & -2 \\ 4 & -1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & -1 \\ 4 & 8 \end{vmatrix}$

$= \underset{\sim}{i}(1+16) - \underset{\sim}{j}(-2+8) + \underset{\sim}{k}(16+4)$

$\therefore \underset{\sim}{A} \times \underset{\sim}{B} = 17\underset{\sim}{i} - 6\underset{\sim}{j} + 20\underset{\sim}{k}$.

8.(a) $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are two vectors in the plane.

$\overrightarrow{PQ} = (3-1)\underset{\sim}{i} + (0+2)\underset{\sim}{j} + (8-5)\underset{\sim}{k}$

$\therefore \overrightarrow{PQ} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

$\overrightarrow{PR} = (1-1)\underset{\sim}{i} + (-4+2)\underset{\sim}{j} + (8-5)\underset{\sim}{k}$

$\therefore \overrightarrow{PR} = -2\underset{\sim}{j} + 3\underset{\sim}{k}$.

(i) $\underset{\sim}{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is a vector perpendicular to the plane.

$\therefore \underset{\sim}{N} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 2 & 3 \\ 0 & -2 & 3 \end{vmatrix}$

$= \underset{\sim}{i}\begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix}$

$= \underset{\sim}{i}(6+6) - \underset{\sim}{j}(6-0) + \underset{\sim}{k}(-4-0)$

$\therefore \underset{\sim}{N} = 12\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}$ is a vector perpendicular to the plane (as is any non-zero multiple).

(ii) Triangle PQR has area, A, given by

$A = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$= \frac{1}{2}|12\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}|$

$= |6\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}|$

$= \sqrt{6^2 + (-3)^2 + (-2)^2}$

$= \sqrt{36+9+4} = \sqrt{49}$.

$\therefore A = 7$.

(iii) $\underset{\sim}{N} = 12\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}$ is normal to the plane.

$\therefore$ The plane has equation

$12x - 6y - 4z = D$.

Since $P(1,-2,5)$ is in the plane

$(12)(1) - (6)(-2) - (4)(5) = D$

$\therefore D = 12 + 12 - 20 = 4$.

$\therefore$ The equation is

$12x - 6y - 4z = 4$

i.e. $6x - 3y - 2z = 2$.

8. (b). Given $P(5,-1,7)$ and $Q(3,0,4)$,

$\underset{\sim}{v} = \overrightarrow{PQ} = (3-5)\underline{i} + (0+1)\underline{j} + (4-7)\underline{k}$

is parallel to the line

$\therefore \underset{\sim}{v} = -2\underline{i} + \underline{j} - 3\underline{k}$.

Choosing P as a point on the line, the line has parametric equations:

$x = 5-2t$; $y = -1+t$; $z = 7-3t$.

(c) Substituting the equations of the line:

$x = 4-3t$; $y = -5+t$; $z = 1+2t$

into the equation of the plane

$$4x - 3y + 5z = 1$$

gives

$$4(4-3t) - 3(-5+t) + 5(1+2t) = 1$$

$$\therefore 16 - 12t + 15 - 3t + 5 + 10t = 1$$

$$\therefore -5t + 36 = 1$$

$$\therefore -5t = -35$$

$$\therefore t = 7.$$

When $t = 7$,

$x = 4 - 21 = -17$,

$y = -5 + 7 = 2$,

$z = 1 + 14 = 15$.

$\therefore$ The point of intersection is $(-17, 2, 15)$.

(d) Given $\underset{\sim}{A} = 3\underline{i} + \underline{j} + 7\underline{k}$, $\underset{\sim}{B} = 2\underline{i} - \underline{j} + \underline{k}$, and $\underset{\sim}{C} = \underline{i} - 6\underline{j}$

$$\underset{\sim}{A} \cdot (\underline{B} \times \underline{C}) = \begin{vmatrix} 3 & 1 & 7 \\ 2 & -1 & 1 \\ 1 & -6 & 0 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 1 \\ -6 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + 7\begin{vmatrix} 2 & -1 \\ 1 & -6 \end{vmatrix}$$

$$= 3(0+6) - (0-1) + 7(-12+1)$$

$$= 18 + 1 - 77$$

$$\therefore \underset{\sim}{A} \cdot (\underline{B} \times \underline{C}) = -58.$$

$\therefore$ The volume, V, is given by

$$V = |\underset{\sim}{A} \cdot (\underline{B} \times \underline{C})|$$

$$= |-58|$$

$\therefore$ Volume is 58.