1. (a) $y = f(x) = |2x - 7|$

(i) Domain : all real x.

Range : $y \geq 0$.

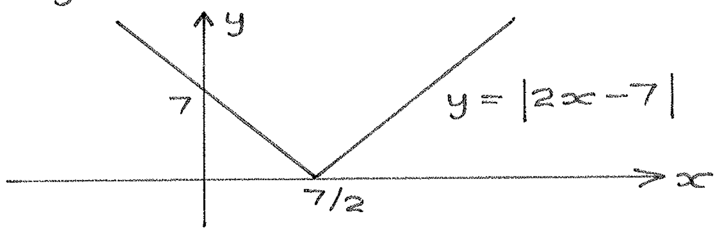
As $y = \begin{cases} 2x-7 & \text{if } 2x-7 \geq 0 \\ -(2x-7) & \text{if } 2x-7 < 0 \end{cases}$

i.e. $y = \begin{cases} 2x-7 & \text{if } x \geq \frac{7}{2} \\ -2x+7 & \text{if } x < \frac{7}{2} \end{cases}$

the sketch consists of the 2 straight lines :

$y = 2x - 7$ for $x \geq \frac{7}{2}$;

$y = -2x + 7$ for $x < \frac{7}{2}$.



When $x = 0$, $y = 7$

When $y = 0$, $x = \frac{7}{2}$.

(ii) Let $f(a) = f(b)$

$\therefore |2a - 7| = |2b - 7|$

$\therefore 2a - 7 = \pm(2b - 7)$

$\therefore 2a - 7 = 2b - 7$ or

$2a - 7 = -2b + 7$

$\therefore 2a = 2b$ or $2a = -2b + 14$

$\therefore a = b$ or $a = -b + 7$.

As $a = b$ is not the only solution, $f(x)$ is not one-to-one

(iii) From the sketch, $f(x)$ is one-to-one for either

$x \geq \frac{7}{2}$ or $x \leq \frac{7}{2}$.

(b) $y = f(x) = \sqrt{8x - 7}$;

$x \geq \frac{7}{8}$ and $y \geq 0$.

(i) Swapping x and y gives

$x = \sqrt{8y - 7}$; $x \geq 0$, $y \geq \frac{7}{8}$.

$\therefore x^2 = 8y - 7$

$\therefore x^2 + 7 = 8y$

$\therefore y = \dfrac{x^2 + 7}{8}$.

$\therefore$ The inverse function is

$y = f^{-1}(x) = \dfrac{x^2 + 7}{8}$

for $x \geq 0$ and $y \geq \frac{7}{8}$.

(ii) The curves $y = f(x)$ and $y = f^{-1}(x)$ intersect when $y = x$, i.e. when

$f^{-1}(x) = x$

$\therefore \dfrac{x^2 + 7}{8} = x$

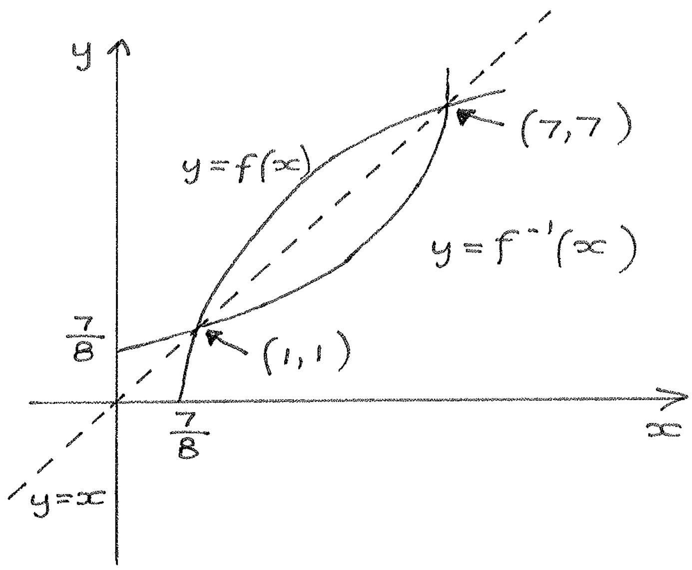
$\therefore x^2 + 7 = 8x$

$\therefore x^2 - 8x + 7 = 0$

$\therefore (x - 1)(x - 7) = 0$

$\therefore x = 1$ and $x = 7$.

Note that $y = \dfrac{x^2 + 7}{8}$ is a concave up parabola.



1.(c) (i) $\lim_{x\to\infty} \dfrac{2x^3-6x^2+5}{3-4x^3}$

$= \lim_{x\to\infty} \dfrac{x^3\left(2-\frac{6}{x}+\frac{5}{x^3}\right)}{x^3\left(\frac{3}{x^3}-4\right)}$

$= \lim_{x\to\infty} \dfrac{2-\frac{6}{x}+\frac{5}{x^3}}{\frac{3}{x^3}-4}$

$= \dfrac{2-0+0}{0-4} = \dfrac{-2}{4} = -\dfrac{1}{2}.$

(ii) $\lim_{x\to 5} \dfrac{x^2-3x-10}{x^2-5x} \quad \left(=\frac{0}{0}\right)$

$= \lim_{x\to 5} \dfrac{(x+2)(x-5)}{x(x-5)}$

$= \lim_{x\to 5} \dfrac{x+2}{x} = \dfrac{5+2}{5} = \dfrac{7}{5}.$

OR (by L'H.R)

$= \lim_{x\to 5} \dfrac{2x-3}{2x-5} = \dfrac{7}{5}.$

(iii) $\lim_{x\to 1} \dfrac{2x^9+4x^5-5x^2-1}{x^4-1}$

$\left(=\frac{0}{0}\right.$ $\therefore$ Use L'H.R$\left.\right)$

$= \lim_{x\to 1} \dfrac{18x^8+20x^4-10x}{4x^3}$

$= \dfrac{18+20-10}{4} = \dfrac{28}{4} = 7.$

(iv) $\lim_{x\to 0} \dfrac{7\sin x+e^{4x}-1}{5x}$

$\left(=\frac{0+1-1}{0}=\frac{0}{0}\right.$ $\therefore$ Use L'H.R$\left.\right)$

$= \lim_{x\to 0} \dfrac{7\cos x+4e^{4x}}{5}$

$= \dfrac{7+4}{5} = \dfrac{11}{5}.$

2.(a) (i) $y = \dfrac{2x+5}{4x-3}$

$\therefore$ By the Quotient Rule,

$\dfrac{dy}{dx} = \dfrac{(4x-3)(2)-4(2x+5)}{(4x-3)^2}$

$= \dfrac{8x-6-8x-20}{(4x-3)^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{-26}{(4x-3)^2}.$

(ii) $y = (x\ln x - x)^8$

Let $u = x\ln x - x$ $\quad\therefore y = u^8$

$\therefore \dfrac{du}{dx} = x\left(\dfrac{1}{x}\right) + \ln x - 1$

(Product Rule)

$\therefore \dfrac{du}{dx} = 1+\ln x - 1 = \ln x.$

and $\dfrac{dy}{du} = 8u^7.$

$\therefore$ By the Chain Rule,

$\dfrac{dy}{dx} = 8u^7 \ln x$

$\therefore \dfrac{dy}{dx} = 8(x\ln x - x)^7 \ln x.$

(iii) $y = (2\sin x - \cos x)e^{2x}$

$\therefore$ By the Product Rule,

$\dfrac{dy}{dx} = (2\sin x - \cos x)(2e^{2x}) + (2\cos x + \sin x)e^{2x}$

$= (4\sin x - 2\cos x + 2\cos x + \sin x)e^{2x}$

$\therefore \dfrac{dy}{dx} = 5\sin x\, e^{2x}.$

(iv) $3x^3 - 7xy - y^4 = 5.$

$\therefore \dfrac{d}{dx}\left[3x^3-7xy-y^4\right] = \dfrac{d}{dx}(5)$

$\therefore 9x^2 - 7\left(x\dfrac{dy}{dx}+y\right) - \dfrac{d}{dx}(y^4) = 0$

(Product Rule)

$\therefore 9x^2 - 7x\dfrac{dy}{dx} - 7y - 4y^3\dfrac{dy}{dx} = 0$

(Chain Rule)

$\therefore 9x^2 - 7y = 7x\dfrac{dy}{dx} + 4y^3\dfrac{dy}{dx}$

$\therefore 9x^2 - 7y = (7x+4y^3)\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{9x^2-7y}{7x+4y^3}.$

2. (b) $y = \dfrac{(9x^2-5)^{2/9}}{(x^5-15x+1)^{1/5}}$

$\therefore \ln y = \ln\left[\dfrac{(9x^2-5)^{2/9}}{(x^5-15x+1)^{1/5}}\right]$

$= \ln\left[(9x^2-5)^{2/9}\right] - \ln\left[(x^5-15x+1)^{1/5}\right]$

$= \frac{2}{9}\ln(9x^2-5) - \frac{1}{5}\ln(x^5-15x+1)$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2}{9}\left(\dfrac{1}{9x^2-5}\right)(18x) - \dfrac{1}{5}\left(\dfrac{1}{x^5-15x+1}\right)(5x^4-15)$

$\therefore \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{4x}{9x^2-5} - \dfrac{x^4-3}{x^5-15x+1}$

$\therefore \dfrac{dy}{dx} = y\left[\dfrac{4x}{9x^2-5} - \dfrac{x^4-3}{x^5-15x+1}\right]$

(c) (i) $y = (2x^2+3)\sin^{-1}x$

By the Product Rule,

$\dfrac{dy}{dx} = (2x^2+3)\left(\dfrac{1}{\sqrt{1-x^2}}\right) + 4x\sin^{-1}x$

$\therefore \dfrac{dy}{dx} = \dfrac{2x^2+3}{\sqrt{1-x^2}} + 4x\sin^{-1}x.$

(ii) $y = \cosh(4\sqrt{x})$

Let $u = 4\sqrt{x}$ $\quad\therefore y = \cosh u$

$\therefore \dfrac{du}{dx} = \dfrac{2}{\sqrt{x}}$; $\dfrac{dy}{du} = \sinh u$

$\therefore$ By the Chain Rule,

$\dfrac{dy}{dx} = \dfrac{2}{\sqrt{x}}\sinh u$

$\therefore \dfrac{dy}{dx} = \dfrac{2\sinh(4\sqrt{x})}{\sqrt{x}}.$

(iii) $y = \displaystyle\int_3^x \frac{e^{2t}-5}{\ln(t+2)}\,dt$

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}\left[\displaystyle\int_3^x \frac{e^{2t}-5}{\ln(t+2)}\,dt\right]$

$\therefore$ By the Fundamental Theorem,

$\dfrac{dy}{dx} = \dfrac{e^{2x}-5}{\ln(x+2)}.$

3. (a) (i) By definition,

$\cosh x = \frac{1}{2}(e^x + e^{-x})$;

$\sinh x = \frac{1}{2}(e^x - e^{-x})$.

(ii) To Prove: For $x > 0$,

$\cosh(\ln x) - \sinh(\ln x) = \dfrac{1}{x}$.

Proof: L.S. $= \cosh(\ln x) - \sinh(\ln x)$

$= \frac{1}{2}(e^{\ln x} + e^{-\ln x}) - \frac{1}{2}(e^{\ln x} - e^{-\ln x})$

$= \frac{1}{2}\left(x + \dfrac{1}{e^{\ln x}}\right) - \frac{1}{2}\left(x - \dfrac{1}{e^{\ln x}}\right)$

$= \frac{1}{2}\left(x + \frac{1}{x} - x + \frac{1}{x}\right)$

$= \frac{1}{2}\left(\frac{2}{x}\right)$

$= \frac{1}{x}$

$=$ R.S, as required.

(b) $y = f(x) = x^3 - 3x^2 - 9x + 6$ for $-4 \le x \le 3$.

$f(x)$ is continuous and differentiable for $-4 \le x \le 3$.

$f'(x) = 3x^2 - 6x - 9$

$= 3(x^2 - 2x - 3)$

$= 3(x+1)(x-3)$

$\therefore f'(x) = 0$ when $x = -1$ and $x = 3$.

$\therefore$ Consider the points: $x = -4$, $x = -1$, and $x = 3$.

$f(-4) = -64 - 48 + 36 + 6 = -70.$

$f(-1) = -1 - 3 + 9 + 6 = 11.$

$f(3) = 27 - 27 - 27 + 6 = -21.$

$\therefore$ The absolute maximum is 11 (when $x = -1$).

The absolute minimum is -70 (when $x = -4$).

3. (c) $y = f(x) = \dfrac{2(x^2-4)}{(x+4)^2}$.

(i) Domain: all x, except $x = -4$.

(ii) Vertical asymptote: $x = -4$.

as $x \to -4$, $y \to \dfrac{24}{0}$ (infinite)

(iii) Symmetry:

$$f(x) = \frac{2(x^2-4)}{(x+4)^2}$$

$$\therefore f(-x) = \frac{2[(-x)^2-4]}{(-x+4)^2}$$

$$= \frac{2(x^2-4)}{(x-4)^2} \neq \begin{cases} f(x) \\ -f(x) \end{cases}$$

$\therefore f(x)$ is neither even nor odd.

(iv) Intercepts:

When $x = 0$, $y = \dfrac{-8}{16} = -\dfrac{1}{2}$.

When $y = 0$, $x^2 - 4 = 0$

$\therefore x^2 = 4 \quad \therefore x = \pm 2$.

(v) As $x \to \pm\infty$,

$$y = \frac{2(x^2-4)}{(x+4)^2} \sim \frac{2x^2}{x^2} = 2.$$

Crossings of $x = 2$ occur when

$$\frac{2(x^2-4)}{(x+4)^2} = 2$$

$$\therefore 2(x^2-4) = 2(x+4)^2$$

$$\therefore x^2 - 4 = (x+4)^2$$

$$\therefore x^2 - 4 = x^2 + 8x + 16$$

$$\therefore -4 = 8x + 16$$

$$\therefore 8x = -20$$

$$\therefore x = \frac{-20}{8} = \frac{-5}{2}.$$

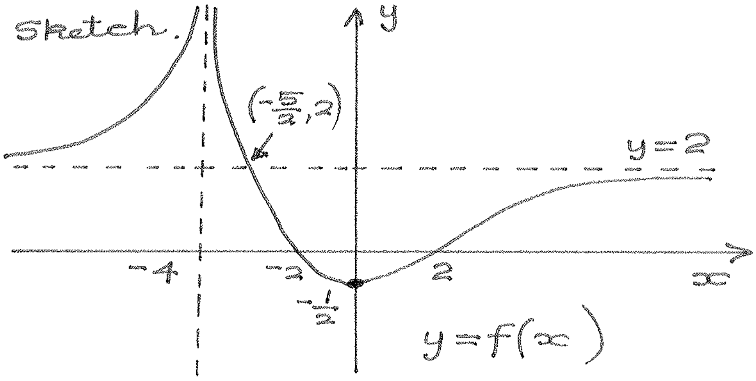
(vi) Sign of y: is the sign of $x^2 - 4$, i.e.

$y < 0$ when $x^2 - 4 < 0$

i.e. $-2 < x < 2$.

$\therefore y > 0$ when $x < -2$, and $x > 2$.

Sketch.



4. (a) (i) $I = \displaystyle\int \frac{x^3}{\sqrt{2x^4-1}}\,dx$

Let $u = 2x^4 - 1$

$$\therefore \frac{du}{dx} = 8x^3$$

$$\therefore du = 8x^3\,dx$$

$$\therefore \frac{1}{8}du = x^3\,dx$$

and $\sqrt{2x^4-1} = \sqrt{u}$

$$\therefore I = \frac{1}{8}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{8}\int u^{-1/2}\,du$$

$$= \frac{1}{8}\left(2u^{1/2}\right) + C$$

$$\therefore I = \frac{1}{4}\sqrt{2x^4-1} + C.$$

(ii) $I = \displaystyle\int (6x+3)e^{2x}\,dx$

Let $u = 6x+3$; $\dfrac{dv}{dx} = e^{2x}$

$\therefore \dfrac{du}{dx} = 6$; $v = \dfrac{1}{2}e^{2x}$

$$\therefore I = (6x+3)\left(\tfrac{1}{2}e^{2x}\right) - \tfrac{1}{2}\int 6e^{2x}\,dx$$

$$= \tfrac{1}{2}(6x+3)e^{2x} - \tfrac{3}{2}e^{2x} + C$$

$$= \tfrac{1}{2}(6x+3-3)e^{2x} + C$$

$$\therefore I = 3xe^{2x} + C.$$

4.(a) (iii) $I = \int \frac{\cos x}{(4-\sin x)^2}\,dx$

Let $u = 4 - \sin x$

$\therefore \frac{du}{dx} = -\cos x$

$\therefore du = -\cos x\,dx$

$\therefore -du = \cos x\,dx$

and $(4-\sin x)^2 = u^2$

$\therefore I = -\int \frac{1}{u^2}\,du$

$= -\int u^{-2}\,du$

$= -\left(-u^{-1}\right) + C$

$= \frac{1}{u} + C$

$\therefore I = \frac{1}{4-\sin x} + C.$

(b) $I = \int_0^1 \frac{1}{x^2+3}\,dx$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$

$(\text{S.I.}3\ ;\ a = \sqrt{3})$

$\therefore I = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}0\right]$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right)$

$\therefore I = \frac{\pi}{6\sqrt{3}}.$

(c) $I = \int \frac{x-9}{x^2-6x+5}\,dx$

As $x^2 - 6x + 5 = (x-1)(x-5)$,

$\frac{x-9}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5}$

$\therefore x - 9 = A(x-5) + B(x-1)$

When $x = 1$, $-8 = -4A$

$\therefore A = 2.$

When $x = 5$, $-4 = 4B$

$\therefore B = -1.$

$\therefore \frac{x-9}{(x-1)(x-5)} = \frac{2}{x-1} - \frac{1}{x-5}$

$\therefore I = 2\ln|x-1| - \ln|x-5| + C$

$= \ln|x-1|^2 - \ln|x-5| + C$

$\therefore I = \ln\left|\frac{(x-1)^2}{x-5}\right| + C.$

5. (a)(i) $I = \int_0^{\sqrt{2}} \frac{4x^3}{\sqrt{x^8+9}}\,dx$

Let $u = x^4$ $\therefore \frac{du}{dx} = 4x^3$

and $x^8 = u^2$

$\therefore \sqrt{x^8+9} = \sqrt{u^2+9}$

Terminals: $x = 0 \Rightarrow u = 0$

$x = \sqrt{2} \Rightarrow u = 4.$

$\therefore I = \int_0^4 \frac{1}{\sqrt{u^2+9}}\,du$

(ii) $I = \left[\ln\left\{u + \sqrt{u^2+9}\right\}\right]_0^4$

$(\text{S.I.}6\ ;\ a = 3)$

$= \ln\left\{4 + \sqrt{16+9}\right\} - \ln\left\{0 + \sqrt{9}\right\}$

$= \ln(4+5) - \ln 3$

$= \ln 9 - \ln 3 = \ln\left(\frac{9}{3}\right)$

$\therefore I = \ln 3.$

(b) The concave up parabola

$y = (x-2)^2 = x^2 - 4x + 4$

and the line $y = 7 - 2x$

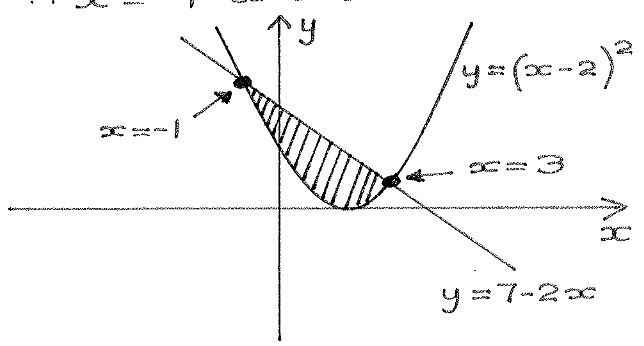
intersect when:

$x^2 - 4x + 4 = 7 - 2x$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x+1)(x-3) = 0$

$\therefore x = -1$ and $x = 3.$



5. (b) (continued)

From the sketch,

$f(x)-g(x) = 7-2x-(x^2-4x+4)$

$= 7-2x-x^2+4x-4$

$= 3+2x-x^2.$

$\therefore$ Area $A = \int_{-1}^{3}(3+2x-x^2)\,dx$

$= \left[3x+x^2-\frac{x^3}{3}\right]_{-1}^{3}$

$= (9+9-9)-(-3+1+\frac{1}{3})$

$= 9+3-1-\frac{1}{3}$

$= 11-\frac{1}{3}$

$\therefore A = \frac{32}{3}.$

(c) $y = 3(1-\sqrt{x})\sqrt{x}$

is defined for $x \geq 0$ only.

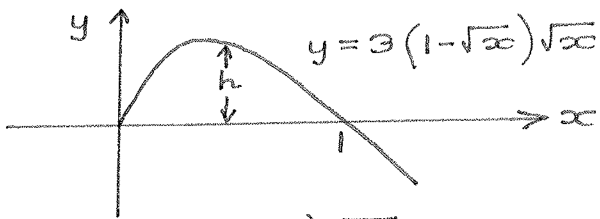
When $y=0$,

$(1-\sqrt{x})\sqrt{x} = 0$

$\therefore x=0$ and $x=1$.

For $0<x<1$, $y>0$.

For $x>1$, $y<0$.



$\therefore h = 3(1-\sqrt{x})\sqrt{x}$

$\therefore h^2 = 9(1-2\sqrt{x}+x)x$

$\therefore h^2 = 9(x-2x^{3/2}+x^2)$

$\therefore$ Volume $V = \pi\int_0^1 h^2\,dx$

$= 9\pi\int_0^1(x-2x^{3/2}+x^2)\,dx$

$= 9\pi\left[\frac{x^2}{2}-\frac{4x^{5/2}}{5}+\frac{x^3}{3}\right]_0^1$

$= 9\pi\left(\frac{1}{2}-\frac{4}{5}+\frac{1}{3}-0\right)$

$= 9\pi\left(\frac{15-24+10}{30}\right) = \frac{9\pi}{30}$

$\therefore V = \frac{3\pi}{10}.$

6. (a)(i) $\frac{dy}{dx} = 9x^2y^{2/3}$;

with $y(2)=1$.

$\therefore dy = 9x^2y^{2/3}\,dx$

$\therefore \frac{1}{y^{2/3}}\,dy = 9x^2\,dx$

(Separated)

$\therefore \int y^{-2/3}\,dy = \int 9x^2\,dx$

$\therefore 3y^{1/3} = 3x^3+C$

$\therefore y^{1/3} = x^3+K$

(where $K = \frac{C}{3}$)

When $x=2$, $y=1$

$\therefore 1^{1/3} = 2^3+K$

$\therefore 1 = 8+K \quad \therefore K=-7$

$\therefore y^{1/3} = x^3-7$

$\therefore$ Solution is:

$y = (x^3-7)^3.$

(ii) $\frac{dy}{dx}+\frac{2y}{x} = 8x+3$; (1)

with $y(1)=8$.

(1) is linear with $p(x) = \frac{2}{x}$.

$\therefore$ Integrating factor $\mu(x)$ is

given by $\mu = e^{\int\frac{2}{x}dx}$

$\therefore \mu = e^{2\ln|x|+C}.$

Choosing $C=0$, and $x>0$,

$\mu = e^{2\ln x} = e^{\ln(x^2)}$

$\therefore \mu = x^2.$

Multiplying (1) by x^2 gives

$x^2\frac{dy}{dx}+2xy = 8x^3+3x^2$

$\therefore \frac{d}{dx}[x^2y] = 8x^3+3x^2$

$\therefore x^2y = \int(8x^3+3x^2)\,dx$

$\therefore x^2y = 2x^4+x^3+K$

6.(a)(ii) (continued)

$\therefore y = \dfrac{2x^4 + x^3 + K}{x^2}$

When $x = 1$, $y = 8$

$\therefore 8 = 2 + 1 + K$

$\therefore K = 5$

$\therefore$ solution is :

$y = \dfrac{2x^4 + x^3 + 5}{x^2}$,

i.e. $y = 2x^2 + x + \dfrac{5}{x^2}$.

(b) (i) $a_n = \dfrac{4n - (-1)^n}{2n}$.

$a_1 = \dfrac{4 - (-1)}{2} = \dfrac{5}{2}$,

$a_2 = \dfrac{8 - 1}{4} = \dfrac{7}{4}$,

$a_3 = \dfrac{12 - (-1)}{6} = \dfrac{13}{6}$,

$a_4 = \dfrac{16 - 1}{8} = \dfrac{15}{8}$.

(ii) In general,

$$a_n = \begin{cases} \dfrac{4n+1}{2n}, & \text{for } n \text{ odd} \\ \dfrac{4n-1}{2n}, & \text{for } n \text{ even.} \end{cases}$$

$$\therefore a_n = \begin{cases} 2 + \dfrac{1}{2n}, & \text{for } n \text{ odd} \\ 2 - \dfrac{1}{2n}, & \text{for } n \text{ even.} \end{cases}$$

As $\lim\limits_{n\to\infty} \left(2 + \dfrac{1}{2n}\right) = 2$,

and $\lim\limits_{n\to\infty} \left(2 - \dfrac{1}{2n}\right) = 2$,

$\lim\limits_{n\to\infty} a_n = 2$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-1)^n}{3^n}$

$\therefore a_n = \dfrac{n(x-1)^n}{3^n}$

$\therefore a_{n+1} = \dfrac{(n+1)(x-1)^{n+1}}{3^{n+1}}$

$\therefore \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$

$= \lim\limits_{n\to\infty} \left|\dfrac{(n+1)(x-1)^{n+1}(3^n)}{(3^{n+1})n(x-1)^n}\right|$

$= \dfrac{|x-1|}{3} \lim\limits_{n\to\infty} \dfrac{n+1}{n}$

$= \dfrac{|x-1|}{3} \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)$

$= \dfrac{|x-1|}{3}$.

For convergence,

$\dfrac{|x-1|}{3} < 1$.

$\therefore |x-1| < 3$.

$\therefore -3 < x - 1 < 3$

i.e. $-2 < x < 4$ is the open interval of convergence.

7.(a)(i) For $f(x) = (1+x)^{3/5}$,

$f(x) = (1+x)^{3/5}$; $f(0) = 1$.

$f'(x) = \frac{3}{5}(1+x)^{-2/5}$; $f'(0) = \frac{3}{5}$.

$f''(x) = \frac{-6}{25}(1+x)^{-7/5}$; $f''(0) = \frac{-6}{25}$.

$\therefore f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \ldots$

$= 1 + \frac{3}{5}x + \frac{1}{2}\left(\frac{-6}{25}x^2\right) + \ldots$

$\therefore (1+x)^{3/5} = 1 + \dfrac{3x}{5} - \dfrac{3x^2}{25} + \ldots$

(ii) When $x = 0.06$,

$(1+0.06)^{3/5} \simeq 1 + \dfrac{3\times 0.06}{5} - \dfrac{3\times(0.06)^2}{25}$

$= 1 + 0.036 - 0.000432$

$\therefore (1.06)^{3/5} \simeq 1.035568$.

(b)(i) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

Replacing x by $2x$ gives

$e^{2x} = 1 + 2x + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \ldots$

$\therefore e^{2x} = 1 + 2x + \dfrac{4x^2}{2!} + \dfrac{8x^3}{3!} + \ldots$

7. (b)(i) (continued)

$$\therefore \frac{e^{2x}-1}{x} = \frac{1}{x}\left[e^{2x}-1\right]$$

$$= \frac{1}{x}\left[\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots\right)-1\right]$$

$$= \frac{1}{x}\left[2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots\right]$$

$$\therefore \frac{e^{2x}-1}{x} = 2+\frac{4x}{2!}+\frac{8x^2}{3!}+\ldots$$

(for all x).

(ii) $$I=\int_0^{0.3}\frac{e^{2x}-1}{x}\,dx$$

$$\simeq \int_0^{0.3}\left(2+\frac{4x}{2!}+\frac{8x^2}{3!}\right)dx$$

$$= \int_0^{0.3}\left(2+2x+\frac{4x^2}{3}\right)dx$$

$$= \left[2x+x^2+\frac{4x^3}{9}\right]_0^{0.3}$$

$$= \left[0.6+0.09+\frac{4\times 0.027}{9}\right]-0$$

$$= 0.69+0.012$$

$$\therefore I \simeq 0.702.$$

(c) $\underset{\sim}{A} = 6\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}$, and

$\underset{\sim}{B} = \underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$.

(i) $$|\underset{\sim}{A}| = \sqrt{6^2+(-2)^2+3^2}$$

$$= \sqrt{36+4+9} = \sqrt{49}.$$

$$\therefore |\underset{\sim}{A}| = 7.$$

$$|\underset{\sim}{B}| = \sqrt{1^2+(-2)^2+2^2}$$

$$= \sqrt{1+4+4} = \sqrt{9}.$$

$$\therefore |\underset{\sim}{B}| = 3.$$

(ii) $$\underset{\sim}{A}\cdot\underset{\sim}{B} = (6\times 1)+(-2\times -2)+(3\times 2)$$

$$= 6+4+6$$

$$\therefore \underset{\sim}{A}\cdot\underset{\sim}{B} = 16.$$

(iii) $$\cos\theta = \frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|}$$

$$= \frac{16}{7\times 3}.$$

$$\therefore \cos\theta = \frac{16}{21}.$$

(iv) $$\underset{\sim}{A}\times\underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 6 & -2 & 3 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} -2 & 3 \\ -2 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 6 & 3 \\ 1 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 6 & -2 \\ 1 & -2 \end{vmatrix}$$

$$= \underset{\sim}{i}(-4+6)-\underset{\sim}{j}(12-3)+\underset{\sim}{k}(-12+2)$$

$$\therefore \underset{\sim}{A}\times\underset{\sim}{B} = 2\underset{\sim}{i}-9\underset{\sim}{j}-10\underset{\sim}{k}.$$

8. (a)(i) Given $P(1,-2,4)$, $Q(-1,1,2)$, and $R(4,-2,6)$, two vectors in the plane are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = (-1-1)\underset{\sim}{i}+(1+2)\underset{\sim}{j}+(2-4)\underset{\sim}{k}$$

$$\therefore \overrightarrow{PQ} = -2\underset{\sim}{i}+3\underset{\sim}{j}-2\underset{\sim}{k}.$$

$$\overrightarrow{PR} = (4-1)\underset{\sim}{i}+(-2+2)\underset{\sim}{j}+(6-4)\underset{\sim}{k}$$

$$\therefore \overrightarrow{PR} = 3\underset{\sim}{i}+2\underset{\sim}{k}.$$

A vector perpendicular to the plane is $\underset{\sim}{N} = \overrightarrow{PQ}\times\overrightarrow{PR}$.

$$\therefore \underset{\sim}{N} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & 3 & -2 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 3 & -2 \\ 0 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & -2 \\ 3 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}(6-0)-\underset{\sim}{j}(-4+6)+\underset{\sim}{k}(0-9)$$

$$\therefore \underset{\sim}{N} = 6\underset{\sim}{i}-2\underset{\sim}{j}-9\underset{\sim}{k}$$

is a vector perpendicular to the plane.

(ii) Area of triangle PQR is

$$\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}|\underset{\sim}{N}|$$

8. (a) (ii) (continued)

As $\underset{\sim}{N} = 6\underset{\sim}{i} - 2\underset{\sim}{j} - 9\underset{\sim}{k}$,

$$|\underset{\sim}{N}| = \sqrt{6^2 + (-2)^2 + (-9)^2}$$

$$= \sqrt{36 + 4 + 81} = \sqrt{121}$$

$\therefore |\underset{\sim}{N}| = 11.$

$\therefore$ Area $= \frac{11}{2}$.

(iii) The equation of the plane is $6x - 2y - 9z = D$.

Since $P(1, -2, 4)$ is a point in the plane,

$$6 - (2 \times -2) - (9 \times 4) = D$$

$\therefore D = 6 + 4 - 36 = -26.$

$\therefore$ The equation is

$$6x - 2y - 9z = -26.$$

(b) Given $P(2, -3, 4)$ and $Q(5, -1, 2)$ are points on the line, $\underset{\sim}{v} = \overrightarrow{PQ}$ is a vector parallel to the line.

$$\therefore \underset{\sim}{v} = (5-2)\underset{\sim}{i} + (-1+3)\underset{\sim}{j} + (2-4)\underset{\sim}{k}$$

$$\therefore \underset{\sim}{v} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}.$$

Using the co-ordinates of P, the parametric equations of the line are:

$$x = 2 + 3t\ ;\ y = -3 + 2t\ ;\ z = 4 - 2t$$

(or equivalent).

(c) The equations of the line

$$x = 2 + 3t\ ;\ y = -3 + 5t\ ;\ z = 1 + 3t$$

substituted into the equation of the plane

$$7x - 2y - 2z = 3$$

give

$$7(2+3t) - 2(-3+5t) - 2(1+3t) = 3$$

$$\therefore 14 + 21t + 6 - 10t - 2 - 6t = 3$$

$$\therefore 5t + 18 = 3$$

$$\therefore 5t = -15$$

$$\therefore t = -3.$$

When $t = -3$,

$x = 2 - 9 = -7$,

$y = -3 - 15 = -18$,

$z = 1 - 9 = -8$.

$\therefore$ The point of intersection is $(-7, -18, -8)$.

(d) Given $\underset{\sim}{A} = 5\underset{\sim}{i} + \underset{\sim}{j} - 6\underset{\sim}{k}$,

$\underset{\sim}{B} = 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$,

and $\underset{\sim}{C} = \underset{\sim}{i} + 7\underset{\sim}{j} - 3\underset{\sim}{k}$,

$$V = |\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C})|$$

where $\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = \begin{vmatrix} 5 & 1 & -6 \\ 2 & -1 & 4 \\ 1 & 7 & -3 \end{vmatrix}$

$$= 5\begin{vmatrix} -1 & 4 \\ 7 & -3 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 \\ 1 & 7 \end{vmatrix}$$

$$= 5(3 - 28) - (-6 - 4) - 6(14 + 1)$$

$$= -125 + 10 - 90$$

$$= -205.$$

$$\therefore V = |-205|$$

$$= 205.$$