1. (a) $y = f(x) = |2x - 9|$

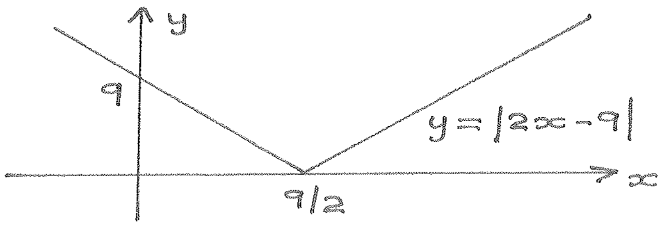
(i) Domain: all real x.

Range: $y \geq 0$.

As $y = \begin{cases} 2x - 9 & \text{if } 2x - 9 \geq 0 \\ -(2x-9) & \text{if } 2x - 9 < 0 \end{cases}$

$y = \begin{cases} 2x - 9 & \text{if } x \geq \frac{9}{2} \\ -2x + 9 & \text{if } x < \frac{9}{2} \end{cases}$,

and the sketch consists of 2 straight lines which meet at $x = \frac{9}{2}$, i.e. when $y = 0$, $x = \frac{9}{2}$.



(ii) Let $f(a) = f(b)$

$\therefore |2a - 9| = |2b - 9|$

$\therefore 2a - 9 = \pm(2b - 9)$

$\therefore 2a - 9 = 2b - 9$

or $2a - 9 = -2b + 9$

$\therefore 2a = 2b$

or $2a = -2b + 18$

$\therefore a = b$ or $a = -b + 9$.

As $a = b$ is not the only solution, $f(x)$ is not one-to-one.

(iii) From the sketch, $f(x)$ is one-to-one for either

$x \geq \frac{9}{2}$ or $x \leq \frac{9}{2}$.

(b) $y = f(x) = \sqrt{5x - 6}$;

$x \geq \frac{6}{5}$, $y \geq 0$.

(i) Swapping x and y gives

$x = \sqrt{5y - 6}$; $x \geq 0, y \geq \frac{6}{5}$.

$\therefore x^2 = 5y - 6$

$\therefore x^2 + 6 = 5y$

$\therefore y = \frac{x^2 + 6}{5}$.

$\therefore y = f^{-1}(x) = \frac{x^2+6}{5}$

for $x \geq 0$, $y \geq \frac{6}{5}$,

is the inverse function.

(ii) $y = f(x)$ and $y = f^{-1}(x)$ intersect when $y = x$, i.e. when $f^{-1}(x) = x$.

$\therefore \frac{x^2 + 6}{5} = x$

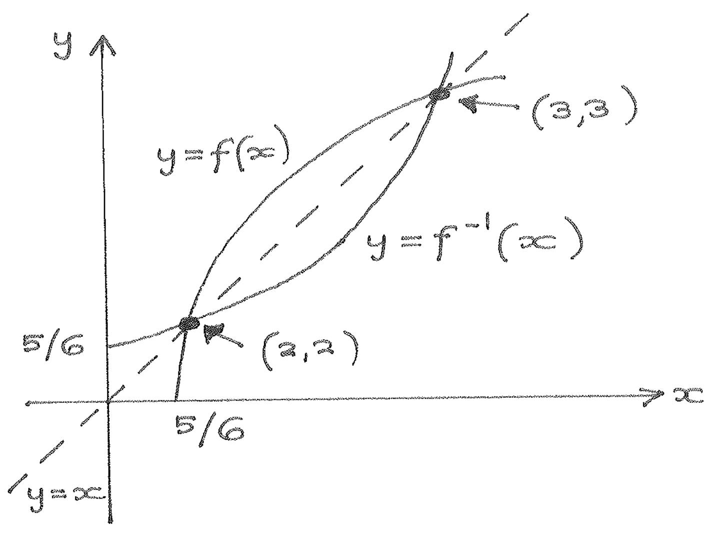
$\therefore x^2 + 6 = 5x$

$\therefore x^2 - 5x + 6 = 0$

$\therefore (x-2)(x-3) = 0$

$\therefore x = 2$ and $x = 3$.

The sketch follows from reflecting the concave up parabola $y = \frac{x^2+6}{5}$ in the line $y = x$.



1. (c) (i) $\lim_{x \to \infty} \dfrac{4x^3 - 3x^2 + 2}{5 - x^3}$

$= \lim_{x \to \infty} \dfrac{x^3\left(4 - \frac{3}{x} + \frac{2}{x^3}\right)}{x^3\left(\frac{5}{x^3} - 1\right)}$

$= \lim_{x \to \infty} \dfrac{4 - \frac{3}{x} + \frac{2}{x^3}}{\frac{5}{x^3} - 1}$

$= \dfrac{4 - 0 + 0}{0 - 1} = \dfrac{4}{-1}$

$= -4.$

(ii) $\lim_{x \to 3} \dfrac{x^2 + x - 12}{x^2 - 4x + 3}$

$\left(= \frac{0}{0}\right) = \lim_{x \to 3} \dfrac{(x-3)(x+4)}{(x-3)(x-1)}$

$= \lim_{x \to 3} \dfrac{x+4}{x-1} = \dfrac{3+4}{3-1}$

$= \dfrac{7}{2}.$

OR (by L'H.R)

$\lim_{x \to 3} \dfrac{2x+1}{2x-4} = \dfrac{6+1}{6-4} = \dfrac{7}{2}.$

(iii) $\lim_{x \to 1} \dfrac{3x^4 + 2x^2 - 6x + 1}{x^5 - 1}$

$\left(= \dfrac{3+2-6+1}{1-1} = \dfrac{0}{0} \therefore \text{Use L'H.R}\right)$

$= \lim_{x \to 1} \dfrac{12x^3 + 4x - 6}{5x^4}$

$= \dfrac{12 + 4 - 6}{5} = \dfrac{10}{5}$

$= 2.$

(iv) $\lim_{x \to 0} \dfrac{4\cos x - e^{5x} - 3}{x - 2\sin x}$

$\left(= \dfrac{4-1-3}{0-0} = \dfrac{0}{0} \therefore \text{Use L'H.R}\right)$

$= \lim_{x \to 0} \dfrac{-4\sin x - 5e^{5x}}{1 - 2\cos x}$

$= \dfrac{0-5}{1-2} = \dfrac{-5}{-1}$

$= 5.$

2. (a) (i) $y = \dfrac{7x-1}{5x+3}$

By the Quotient Rule,

$\dfrac{dy}{dx} = \dfrac{(5x+3)(7) - (7x-1)(5)}{(5x+3)^2}$

$= \dfrac{35x + 21 - 35x + 5}{(5x+3)^2}$

$= \dfrac{26}{(5x+3)^2}.$

(ii) $y = (x\cos x - \sin x)^3$

Let $u = x\cos x - \sin x \therefore y = u^3$

$\therefore \dfrac{du}{dx} = -x\sin x + \cos x - \cos x$

(Product Rule)

$\therefore \dfrac{du}{dx} = -x\sin x$

and $\dfrac{dy}{du} = 3u^2.$

$\therefore$ By the Chain Rule,

$\dfrac{dy}{dx} = -3x\sin x\, u^2$

$\therefore \dfrac{dy}{dx} = -3x\sin x(x\cos x - \sin x)^2.$

(iii) $y = (x^2 - x + 5)e^{2x}$

$\therefore$ By the Product Rule,

$\dfrac{dy}{dx} = (x^2 - x + 5)(2e^{2x})$

$+ (2x - 1)e^{2x}$

$= (2x^2 - 2x + 10)e^{2x}$

$+ (2x - 1)e^{2x}$

$= (2x^2 - 2x + 10 + 2x - 1)e^{2x}$

$= (2x^2 + 9)e^{2x}.$

2.(a) (iv) $3x^4 - 5xy - 2y^3 = 10$

$$\therefore \frac{d}{dx}\left[3x^4 - 5xy - 2y^3\right] = \frac{d}{dx}(10)$$

$$\therefore 12x^3 - 5\left(x\frac{dy}{dx} + y\right) - \frac{d}{dx}(2y^3) = 0$$

(Product Rule)

$$\therefore 12x^3 - 5x\frac{dy}{dx} - 5y - 6y^2\frac{dy}{dx} = 0$$

(Chain Rule)

$$\therefore 12x^3 - 5y = 5x\frac{dy}{dx} + 6y^2\frac{dy}{dx}$$

$$\therefore 12x^3 - 5y = (5x + 6y^2)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{12x^3 - 5y}{5x + 6y^2}.$$

(b) $y = \dfrac{(4x^2-3)^{5/8}}{(2x^3-6x+3)^{1/6}}$

$$\therefore \ln y = \ln\left[\frac{(4x^2-3)^{5/8}}{(2x^3-6x+3)^{1/6}}\right]$$

$$= \ln\left[(4x^2-3)^{5/8}\right] - \ln\left[(2x^3-6x+3)^{1/6}\right]$$

$$= \frac{5}{8}\ln(4x^2-3) - \frac{1}{6}\ln(2x^3-6x+3)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{5}{8}\left(\frac{1}{4x^2-3}\right)(8x) - \frac{1}{6}\left(\frac{1}{2x^3-6x+3}\right)(6x^2-6)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{5x}{4x^2-3} - \frac{x^2-1}{2x^3-6x+3}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{5x}{4x^2-3} - \frac{x^2-1}{2x^3-6x+3}\right]$$

(c) (i) $y = \tan^{-1}(2\sqrt{x})$

Let $u = 2\sqrt{x} = 2x^{1/2}$

$\therefore y = \tan^{-1} u$

$$\therefore \frac{du}{dx} = x^{-1/2} = \frac{1}{\sqrt{x}}$$

and $\dfrac{dy}{du} = \dfrac{1}{1+u^2}$

$\therefore$ By the Chain Rule,

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{1+u^2}\right)$$

As $u^2 = (2\sqrt{x})^2 = 4x$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}\,(1+4x)}.$$

(ii) $y = (4x^3+2)\cosh x$

By the Product Rule,

$$\frac{dy}{dx} = (4x^3+2)\sinh x + 12x^2\cosh x.$$

(iii) $y = \displaystyle\int_4^x \frac{\ln t - 5\sin t}{e^{4t}+1}\,dt$

By the Fundamental Theorem,

$$\frac{dy}{dx} = \frac{\ln x - 5\sin x}{e^{4x}+1}.$$

3.(a) (i) By definition,

$$\cosh x = \frac{1}{2}(e^x + e^{-x});$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

(ii) To Prove:

$\ln(\cosh x - \sinh x) = -x.$

Proof: L.S. $= \ln(\cosh x - \sinh x)$

$$= \ln\left[\tfrac{1}{2}(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x})\right]$$

$$= \ln\left[\tfrac{1}{2}(e^x + e^{-x} - e^x + e^{-x})\right]$$

$$= \ln\left[\tfrac{1}{2}(2e^{-x})\right]$$

$$= \ln(e^{-x})$$

$$= -x$$

= R.S, as required.

3. (b) $y = f(x) = 2x^3 + 3x^2 - 12x + 5$ for $-3 \leq x \leq 3$.

$f(x)$ is continuous and differentiable for $-3 \leq x \leq 3$.

$$f'(x) = 6x^2 + 6x - 12$$
$$= 6(x^2 + x - 2)$$
$$= 6(x+2)(x-1)$$

$\therefore f'(x) = 0$ when $x = -2$ and $x = 1$.

$\therefore$ Absolute extrema can occur at $x = -3, -2, 1, 3$.

$f(-3) = -54 + 27 + 36 + 5 = 14$.

$f(-2) = -16 + 12 + 24 + 5 = 25$.

$f(1) = 2 + 3 - 12 + 5 = -2$.

$f(3) = 54 + 27 - 36 + 5 = 50$.

$\therefore$ The absolute maximum is 50 (when $x = 3$).

The absolute minimum is -2 (when $x = 1$).

(c) $y = f(x) = \dfrac{8(x^2-1)}{(x-2)^2}$

(i) Domain: all x, except $x = 2$.

(ii) Vertical asymptote: $x = 2$

(as $x \to 2$, $y \to \frac{24}{0}$)

(iii) Symmetry:

$$f(x) = \frac{8(x^2-1)}{(x-2)^2}$$

$$\therefore f(-x) = \frac{8[(-x)^2-1]}{(-x-2)^2}$$

$$\therefore f(-x) = \frac{8(x^2-1)}{(-x-2)^2}$$

$$\therefore f(-x) \neq \begin{cases} f(x) \\ -f(x) \end{cases}$$

$\therefore f(x)$ is neither even nor odd.

(iv) Intercepts:

When $x = 0$, $y = \frac{-8}{4} = -2$.

When $y = 0$, $x^2 - 1 = 0$

$\therefore x^2 = 1 \quad \therefore x = \pm 1$.

(v) As $x \to \pm\infty$,

$$y = \frac{8(x^2-1)}{(x-2)^2} \sim \frac{8x^2}{x^2} = 8.$$

Crossings of $y = 8$ occur when $\dfrac{8(x^2-1)}{(x-2)^2} = 8$

$$\therefore 8(x^2-1) = 8(x-2)^2$$
$$\therefore x^2 - 1 = (x-2)^2$$
$$\therefore x^2 - 1 = x^2 - 4x + 4$$
$$\therefore -1 = -4x + 4$$
$$\therefore 4x = 5$$
$$\therefore x = \frac{5}{4}.$$

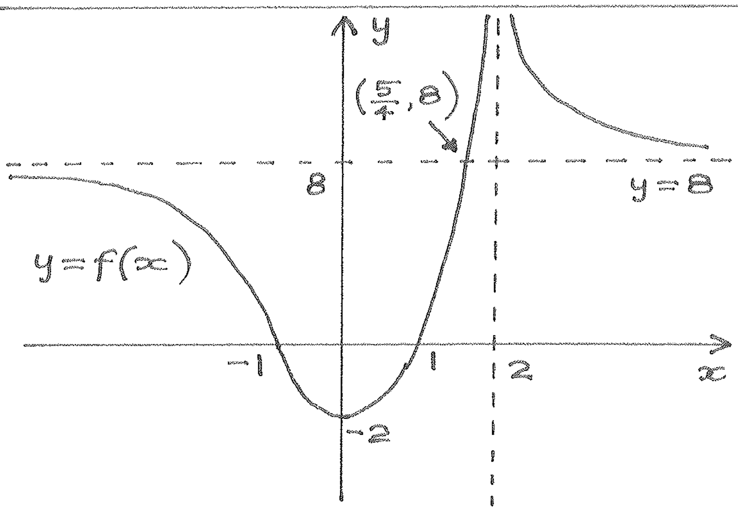
(vi) Sign of y: is the sign of $x^2 - 1$, i.e.

$y > 0$ when $x^2 - 1 > 0$

i.e. $x < -1$ and $x > 1$

$y < 0$ when $x^2 - 1 < 0$

i.e. $-1 < x < 1$.



4.(a)(i) $I = \int \frac{x^2}{(2x^3+5)^2}\, dx$

Let $u = 2x^3 + 5$

$\therefore \frac{du}{dx} = 6x^2$

$\therefore du = 6x^2\, dx$

$\therefore \frac{1}{6} du = x^2\, dx$

and $(2x^3+5)^2 = u^2$

$\therefore I = \frac{1}{6} \int \frac{1}{u^2}\, du$

$= \frac{1}{6} \int u^{-2}\, du$

$= \frac{1}{6}(-u^{-1}) + C$

$= \frac{-1}{6u} + C$

$\therefore I = \frac{-1}{6(2x^3+5)} + C.$

(ii) $I = \int (6x+7)e^{6x}\, dx$

Let $u = 6x+7$; $\frac{dv}{dx} = e^{6x}$

$\therefore \frac{du}{dx} = 6$; $v = \frac{1}{6} e^{6x}$

$\therefore I = (6x+7)(\frac{1}{6}e^{6x}) - \frac{1}{6}\int 6e^{6x} dx$

$= \frac{1}{6}(6x+7)e^{6x} - \int e^{6x}\, dx$

$= \frac{1}{6}(6x+7)e^{6x} - \frac{1}{6} e^{6x} + C$

$= \frac{1}{6}(6x+7-1)e^{6x} + C$

$\therefore I = (x+1)e^{6x} + C.$

(iii) $I = \int \frac{\sin x}{\sqrt{5-2\cos x}}\, dx$

Let $u = 5 - 2\cos x$

$\therefore \frac{du}{dx} = 2\sin x$

$\therefore du = 2\sin x\, dx$

$\therefore \frac{1}{2} du = \sin x\, dx$

and $\sqrt{5-2\cos x} = \sqrt{u}$

$\therefore I = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du$

$= \frac{1}{2}\int u^{-1/2} du$

$= \frac{1}{2}(2u^{1/2}) + C$

$= \sqrt{u} + C$

$\therefore I = \sqrt{5-2\cos x} + C$

(b) $I = \int_0^8 \frac{1}{\sqrt{x^2+36}}\, dx$

$= \left[\ln\{x + \sqrt{x^2+36}\}\right]_0^8$

(S.I.6 ; $a = 6$)

$= \ln\{8 + \sqrt{64+36}\} - \ln\{0 + \sqrt{36}\}$

$= \ln(8+10) - \ln 6$

$= \ln 18 - \ln 6 = \ln\left(\frac{18}{6}\right)$

$\therefore I = \ln 3.$

(c) $I = \int \frac{3x+5}{x^2+2x-3}\, dx$

As $x^2+2x-3 = (x+3)(x-1)$,

$\frac{3x+5}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$

$\therefore 3x+5 = A(x-1) + B(x+3)$

When $x = -3$, $-4 = -4A$

$\therefore A = 1.$

When $x = 1$, $8 = 4B$

$\therefore B = 2.$

$\therefore \frac{3x+5}{(x+3)(x-1)} = \frac{1}{x+3} + \frac{2}{x-1}$

$\therefore I = \ln|x+3| + 2\ln|x-1| + C$

$= \ln|x+3| + \ln|x-1|^2 + C$

$= \ln|(x+3)(x-1)^2| + C.$

5. (a) (i) $I = \int_{\sqrt{2}}^{2} \frac{2x}{\sqrt{16-x^4}}\, dx$

Let $u = x^2 \quad \therefore \frac{du}{dx} = 2x$

$\therefore \quad du = 2x\, dx$

and, as $u^2 = x^4$,

$\sqrt{16-x^4} = \sqrt{16-u^2}$.

Terminals : $x = \sqrt{2} \Rightarrow u = 2$

$x = 2 \Rightarrow u = 4$.

$\therefore I = \int_{2}^{4} \frac{1}{\sqrt{16-u^2}}\, du$

(ii) $I = \left[\sin^{-1}\left(\frac{u}{4}\right) \right]_2^4$

(S.I.4 ; $a = 4$)

$\therefore I = \sin^{-1} 1 - \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi - \pi}{6}$

$= \frac{2\pi}{6}$

$\therefore I = \frac{\pi}{3}$.

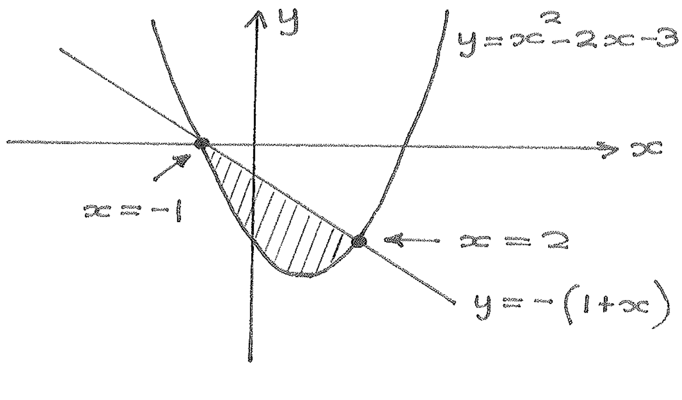
(b) The concave up parabola $y = x^2 - 2x - 3$ and the line $y = -(1+x)$ intersect when

$x^2 - 2x - 3 = -1 - x$

$\therefore x^2 - x - 2 = 0$

$\therefore (x+1)(x-2) = 0$.

$\therefore x = -1$ and $x = 2$.



From the sketch,

$f(x) - g(x) = -(1+x) - (x^2 - 2x - 3)$

$= -1 - x - x^2 + 2x + 3$

$= 2 + x - x^2$.

$\therefore$ Area $A = \int_{-1}^{2} (2 + x - x^2)\, dx$

$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2}$

$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$

$= 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$

$= 8 - \frac{9}{3} - \frac{1}{2}$

$= 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2}$

$\therefore \quad A = \frac{9}{2}$.

(c) $y = (4 - x^2)\sqrt{x}$

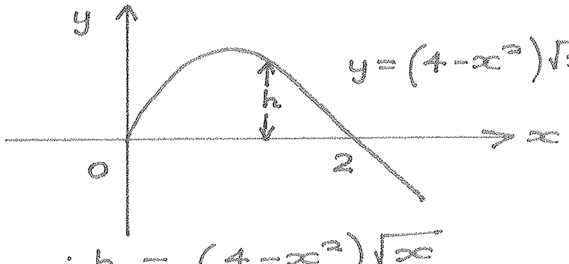
is defined for $x \geq 0$ only.

When $y = 0$, $(4 - x^2)\sqrt{x} = 0$

$\therefore x = 0$ and $x = 2$.

For $0 < x < 2$, $y > 0$

For $x > 2$, $y < 0$.



$\therefore h = (4 - x^2)\sqrt{x}$

$\therefore h^2 = (4 - x^2)^2 x$

$= (16 - 8x^2 + x^4)x$

$\therefore h^2 = 16x - 8x^3 + x^5$

$\therefore \quad V = \pi \int_0^2 (16x - 8x^3 + x^5)\, dx$

$= \pi \left[8x^2 - 2x^4 + \frac{x^6}{6} \right]_0^2$

$= \pi \left[\left(32 - 32 + \frac{64}{6}\right) - 0 \right]$

$= \frac{64\pi}{6}$

$\therefore V = \frac{32\pi}{3}$.

6. (a) (i) $\frac{dy}{dx} = 16xy^{3/4}$;

with $y(2) = 1$.

$\therefore dy = 16xy^{3/4}dx$

$\therefore \quad y^{-3/4}dy = 16x\,dx$

(Separable) $(y \neq 0)$

$\therefore \int y^{-3/4}dy = \int 16x\,dx$

$\therefore \ 4y^{1/4} = 8x^2 + C$

$\therefore y^{1/4} = 2x^2 + K$

(where $K = \frac{C}{4}$)

When $x = 2$, $y = 1$

$\therefore \ 1^{1/4} = 8 + K$

$\therefore \ K = 1 - 8 = -7.$

$\therefore \ y^{1/4} = 2x^2 - 7$

$\therefore$ solution is

$y = (2x^2 - 7)^4.$

(ii) $\frac{dy}{dx} + \frac{y}{x} = 4x^2 - 9x + 4$ ①

with $y(1) = 2$.

① is linear with $p(x) = \frac{1}{x}$.

$\therefore$ The integrating factor is

$\mu = e^{\int \frac{1}{x}dx} = e^{\ln|x| + C}$

choosing $C = 0$ and $x > 0$,

$\mu = e^{\ln x}$

$\therefore \mu = x$.

Multiplying ① by x gives

$x\frac{dy}{dx} + y = 4x^3 - 9x^2 + 4x$

$\therefore \frac{d}{dx}[xy] = 4x^3 - 9x^2 + 4x$

$\therefore xy = \int (4x^3 - 9x^2 + 4x)dx$

$\therefore xy = x^4 - 3x^3 + 2x^2 + K$

$\therefore y = \frac{x^4 - 3x^3 + 2x^2 + K}{x}$

When $x = 1$, $y = 2$

$\therefore 2 = \frac{1 - 3 + 2 + K}{1}$

$\therefore K = 2.$

$\therefore$ solution is

$y = \frac{x^4 - 3x^3 + 2x^2 + 2}{x}$

i.e. $y = x^3 - 3x^2 + 2x + \frac{2}{x}$.

(b) (i) $a_n = \frac{6n + (-1)^n}{n}$

$\therefore a_1 = \frac{6 - 1}{1} = 5,$

$a_2 = \frac{12 + 1}{2} = \frac{13}{2},$

$a_3 = \frac{18 - 1}{3} = \frac{17}{3},$

$a_4 = \frac{24 + 1}{4} = \frac{25}{4}.$

(ii) In general,

$a_n = \begin{cases} \frac{6n - 1}{n} & \text{for } n \text{ odd} \\ \frac{6n + 1}{n} & \text{for } n \text{ even.} \end{cases}$

As $\lim\limits_{n\to\infty} \frac{6n - 1}{n} = \lim\limits_{n\to\infty} (6 - \frac{1}{n}) = 6,$

and $\lim\limits_{n\to\infty} \frac{6n + 1}{n} = \lim\limits_{n\to\infty} (6 + \frac{1}{n}) = 6,$

$\lim\limits_{n\to\infty} a_n = 6.$

(c) $\sum\limits_{n=2}^{\infty} \frac{(x - 6)^n}{(n - 1)2^n}$

$\therefore a_n = \frac{(x - 6)^n}{(n - 1)2^n}$; $a_{n+1} = \frac{(x - 6)^{n+1}}{(n)2^{n+1}}$

$\therefore \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(x - 6)^{n+1}(n - 1)(2^n)}{(n)(2^{n+1})(x - 6)^n}\right|$

$= \frac{|x - 6|}{2} \lim\limits_{n\to\infty} \frac{n - 1}{n} = \frac{|x - 6|}{2}.$

For convergence, $\frac{|x - 6|}{2} < 1$

$\therefore |x - 6| < 2$

$\therefore -2 < x - 6 < 2$

$\therefore \quad 4 < x < 8.$

7. (a) (i) For $f(x) = (1+x)^{-1/3}$

$f(x) = (1+x)^{-1/3}$; $f(0) = 1$.

$f'(x) = -\frac{1}{3}(1+x)^{-4/3}$; $f'(0) = -\frac{1}{3}$.

$f''(x) = \frac{4}{9}(1+x)^{-7/3}$; $f''(0) = \frac{4}{9}$.

$\therefore f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \ldots$

$= 1 - \frac{1}{3}x + \frac{1}{2}\left(\frac{4}{9}\right)x^2 + \ldots$

$\therefore (1+x)^{-1/3} = 1 - \frac{x}{3} + \frac{2x^2}{9} + \ldots$

(ii) When $x = 0.06$,

$(1+0.06)^{-1/3} \simeq 1 - \frac{0.06}{3} + \frac{2(0.06)^2}{9}$

$= 1 - 0.02 + 0.0008$.

$\therefore (1.06)^{-1/3} \simeq 0.9808$.

(b) (i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

$\therefore \frac{x - \sin x}{x^3} = \frac{1}{x^3}\left[x - \sin x\right]$

$= \frac{1}{x^3}\left[x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)\right]$

$= \frac{1}{x^3}\left[x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right]$

$= \frac{1}{x^3}\left[\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \ldots\right]$

$\therefore \frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \ldots$

(ii) $I = \int_0^1 \frac{x - \sin x}{x^3}\,dx$

$\simeq \int_0^1 \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!}\right)dx$

$= \int_0^1 \left(\frac{1}{6} - \frac{x^2}{120} + \frac{x^4}{5040}\right)dx$

$= \left[\frac{x}{6} - \frac{x^3}{360} + \frac{x^5}{25200}\right]_0^1$

$= \left(\frac{1}{6} - \frac{1}{360} + \frac{1}{25200}\right) - 0$

$= \frac{4200 - 70 + 1}{25200}$

$\therefore I \simeq 0.1639$.

(c) $\underset{\sim}{A} = 4\underset{\sim}{i} - 8\underset{\sim}{j} + \underset{\sim}{k}$, and

$\underset{\sim}{B} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

(i) $|\underset{\sim}{A}| = \sqrt{4^2 + (-8)^2 + 1^2}$

$= \sqrt{16 + 64 + 1} = \sqrt{81}$

$\therefore |\underset{\sim}{A}| = 9$.

$|\underset{\sim}{B}| = \sqrt{2^2 + (-1)^2 + 2^2}$

$= \sqrt{4 + 1 + 4} = \sqrt{9}$

$\therefore |\underset{\sim}{B}| = 3$.

(ii) $\underset{\sim}{A} \cdot \underset{\sim}{B} = (4 \times 2) + (-8 \times -1) + (1 \times 2)$

$= 8 + 8 + 2$

$\therefore \underset{\sim}{A} \cdot \underset{\sim}{B} = 18$.

(iii) $\cos\theta = \frac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|}$

$= \frac{18}{9 \times 3} = \frac{18}{27}$

$\therefore \cos\theta = \frac{2}{3}$.

(iv) $\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & -8 & 1 \\ 2 & -1 & 2 \end{vmatrix}$

$= \underset{\sim}{i}\begin{vmatrix} -8 & 1 \\ -1 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 4 & -8 \\ 2 & -1 \end{vmatrix}$

$= \underset{\sim}{i}(-16+1) - \underset{\sim}{j}(8-2) + \underset{\sim}{k}(-4+16)$

$\therefore \underset{\sim}{A} \times \underset{\sim}{B} = -15\underset{\sim}{i} - 6\underset{\sim}{j} + 12\underset{\sim}{k}$.

8. (a) (i) Given $P(0,-3,4)$, $Q(2,-1,7)$ and $R(0,-5,7)$, two vectors in the plane are $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$\overrightarrow{PQ} = (2-0)\underset{\sim}{i} + (-1+3)\underset{\sim}{j} + (7-4)\underset{\sim}{k}$

$\therefore \overrightarrow{PQ} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

$\overrightarrow{PR} = (0-0)\underset{\sim}{i} + (-5+3)\underset{\sim}{j} + (7-4)\underset{\sim}{k}$

$\therefore \overrightarrow{PR} = -2\underset{\sim}{j} + 3\underset{\sim}{k}$.

Let $\underset{\sim}{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$

8.(a)(i) (continued)

$$\therefore \underset{\sim}{N} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 2 & 3 \\ 0 & -2 & 3 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix}$$

$$= \underset{\sim}{i}(6+6) - \underset{\sim}{j}(6-0) + \underset{\sim}{k}(-4-0)$$

$\therefore \underset{\sim}{N} = 12\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}$ is a vector perpendicular to the plane.

(ii) Area of triangle PQR is

$$\tfrac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \tfrac{1}{2}|\underset{\sim}{N}|$$

$$= \tfrac{1}{2}|12\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}|$$

$$= |6\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}|$$

$$= \sqrt{6^2 + (-3)^2 + (-2)^2}$$

$$= \sqrt{36+9+4} = \sqrt{49}$$

$\therefore$ Area $= 7$.

(iii) The plane has equation

$12x - 6y - 4z = D$.

As $P(0,-3,4)$ is in the plane,

$0 + 18 - 16 = D \quad \therefore D = 2$

$\therefore$ The equation is

$12x - 6y - 4z = 2$

i.e. $6x - 3y - 2z = 1$.

(b) For $P(5,-1,3)$ and $Q(9,4,-3)$

$\underset{\sim}{v} = \overrightarrow{PQ}$ is a vector parallel to the line.

$$\therefore \underset{\sim}{v} = (9-5)\underset{\sim}{i} + (4+1)\underset{\sim}{j} + (-3-3)\underset{\sim}{k}$$

$$\therefore \underset{\sim}{v} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k}.$$

Using the co-ordinates of P, the line has parametric equations:

$x = 5+4t$; $y = -1+5t$; $z = 3-6t$.

(c) Substituting the equations of the line:

$x = 4-3t$; $y = -3+7t$; $z = 1+4t$

into the plane equation:

$5x + 2y - z = 23$

gives

$$5(4-3t) + 2(-3+7t) - (1+4t) = 23$$

$$\therefore 20 - 15t - 6 + 14t - 1 - 4t = 23$$

$$\therefore -5t + 13 = 23$$

$$\therefore -5t = 10$$

$$\therefore t = -2.$$

When $t = -2$,

$x = 4+6 = 10$,

$y = -3-14 = -17$,

$z = 1-8 = -7$.

$\therefore$ The point of intersection is $(10,-17,-7)$.

(d) For $\underset{\sim}{A} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$,

$\underset{\sim}{B} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 7\underset{\sim}{k}$,

and $\underset{\sim}{C} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - \underset{\sim}{k}$,

$V = |\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C})|$, where

$$\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = \begin{vmatrix} 3 & 2 & -1 \\ 4 & -2 & 7 \\ 3 & 6 & -1 \end{vmatrix}$$

$$= 3\begin{vmatrix} -2 & 7 \\ 6 & -1 \end{vmatrix} - 2\begin{vmatrix} 4 & 7 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 4 & -2 \\ 3 & 6 \end{vmatrix}$$

$$= 3(2-42) - 2(-4-21) - (24+6)$$

$$= -120 + 50 - 30$$

$$\therefore \underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = -100$$

$$\therefore V = |-100|$$

$\therefore$ Volume is 100.