1. (a) $y = f(x) = |2x+5|$.

(i) <u>Domain</u> : all real $x$

<u>Range</u> : $y \geq 0$

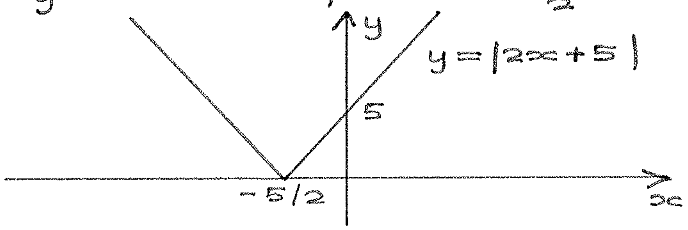
As $y = \begin{cases} 2x+5, & \text{if } 2x+5 \geq 0 \\ -(2x+5), & \text{if } 2x+5 < 0 \end{cases}$

i.e. $y = \begin{cases} 2x+5, & \text{if } x \geq -\frac{5}{2} \\ -2x-5, & \text{if } x < -\frac{5}{2} \end{cases}$

the sketch consists of 2 straight lines:

$y = 2x+5$ for $x \geq -\frac{5}{2}$ ;

$y = -2x-5$ for $x < -\frac{5}{2}$ .




When $x = 0$, $y = 5$.

When $y = 0$, $x = -\frac{5}{2}$ .

(ii) Let $f(a) = f(b)$

$\therefore |2a+5| = |2b+5|$

$\therefore 2a+5 = \pm(2b+5)$

$\therefore 2a+5 = 2b+5$,

or $2a+5 = -2b-5$

$\therefore 2a = 2b$ or $2a = -2b-10$

$\therefore a = b$ or $a = -b-5$.

As $a = b$ is not the only solution, $f(x)$ is not one-to-one.

(iii) From the sketch, $f(x)$ is one-to-one for either

$x \geq -\frac{5}{2}$ or $x \leq -\frac{5}{2}$ .

(b) $y = f(x) = \sqrt{7x-12}$ ;

$x \geq \frac{12}{7}$ and $y \geq 0$

(i) Swapping $x$ and $y$ gives

$x = \sqrt{7y-12}$ ; $y \geq \frac{12}{7}$, $x \geq 0$

$\therefore x^2 = 7y - 12$

$\therefore x^2 + 12 = 7y$

$\therefore y = f^{-1}(x) = \dfrac{x^2+12}{7}$ ;

$x \geq 0$ and $y \geq \frac{12}{7}$

is the inverse function.

(ii) The curves $y = f(x)$ and $y = f^{-1}(x)$ intersect when $y = x$

i.e. when $f^{-1}(x) = x$

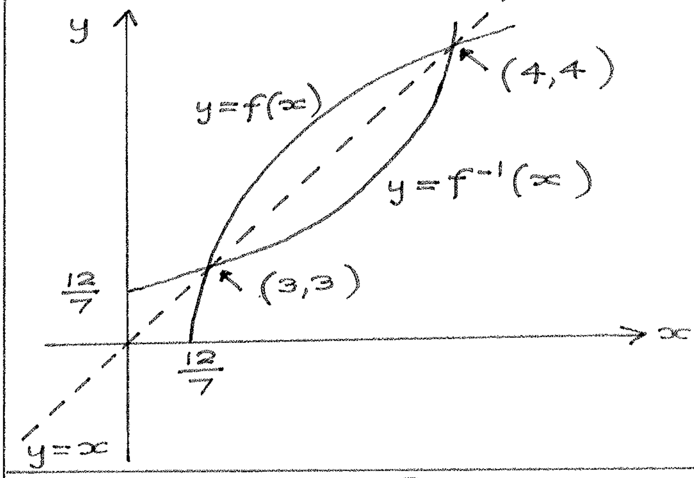
$\therefore \dfrac{x^2+12}{7} = x$

$\therefore x^2 + 12 = 7x$

$\therefore x^2 - 7x + 12 = 0$

$\therefore (x-3)(x-4) = 0$

$\therefore x = 3$ and $x = 4$.




(c) (i) $\lim\limits_{x \to \infty} \dfrac{5x^3 - 4x^2 + 2}{2x^3 + 9}$

$= \lim\limits_{x \to \infty} \dfrac{x^3\left(5 - \frac{4}{x} + \frac{2}{x^3}\right)}{x^3\left(2 + \frac{9}{x^3}\right)}$

$= \lim\limits_{x \to \infty} \dfrac{5 - \frac{4}{x} + \frac{2}{x^3}}{2 + \frac{9}{x^3}}$

$= \dfrac{5 - 0 + 0}{2 + 0} = \dfrac{5}{2}$ .

(ii) $\lim\limits_{x \to 6} \dfrac{x^2 - 4x - 12}{x^2 - 36}$ $\left(= \frac{0}{0}\right)$

$= \lim\limits_{x \to 6} \dfrac{(x-6)(x+2)}{(x-6)(x+6)}$

$= \lim\limits_{x \to 6} \dfrac{x+2}{x+6}$

$= \dfrac{8}{12} = \dfrac{2}{3}$

<u>OR</u> (by L'H.R)

$\lim\limits_{x \to 6} \dfrac{2x-4}{2x} = \dfrac{8}{12} = \dfrac{2}{3}$ .

1. (c) (iii) $\lim\limits_{x\to 1} \dfrac{x^7+7x^5-3x-5}{x^3-1}$

$\left(=\dfrac{0}{0} \quad \therefore \text{ Use L'H.R.}\right)$

$$= \lim_{x\to 1} \frac{7x^6+35x^4-3}{3x^2}$$

$$= \frac{7+35-3}{3} = \frac{39}{3} = 13.$$

(iv) $\lim\limits_{x\to 0} \dfrac{5\cos x - 3e^{2x} - 2}{3x}$

$\left(= \dfrac{5-3-2}{0} = \dfrac{0}{0} \quad \therefore \text{ Use L'H.R}\right)$

$$= \lim_{x\to 0} \frac{-5\sin x - 6e^{2x}}{3}$$

$$= \frac{0-6}{3} = -2.$$

2. (a) (i) $y = \dfrac{3x-1}{5x-2}$

$\therefore$ By the Quotient Rule,

$$\frac{dy}{dx} = \frac{(5x-2)(3) - 5(3x-1)}{(5x-2)^2}$$

$$= \frac{15x - 6 - 15x + 5}{(5x-2)^2}$$

$$= \frac{-1}{(5x-2)^2}.$$

(ii) $y = (x\cos x - \sin x)^6$

Let $u = x\cos x - \sin x$

$\therefore y = u^6$

$$\therefore \frac{du}{dx} = x(-\sin x) + \cos x - \cos x$$

(Product Rule)

$$\therefore \frac{du}{dx} = -x\sin x$$

and $\dfrac{dy}{du} = 6u^5$

$\therefore$ By the Chain Rule,

$$\frac{dy}{dx} = (6u^5)(-x\sin x)$$

$$= -6x\sin x\,(x\cos x - \sin x)^5.$$

(iii) $y = (3\cos x + \sin x)e^{3x}$

By the Product Rule,

$$\frac{dy}{dx} = (3\cos x + \sin x)(3e^{3x}) + (-3\sin x + \cos x)(e^{3x})$$

$$\therefore \frac{dy}{dx} = (9\cos x + 3\sin x - 3\sin x + \cos x)e^{3x}$$

$$= 10\cos x\, e^{3x}.$$

(iv) $3x^2 - 5xy - 2y^3 = 8$

$$\therefore \frac{d}{dx}\left[3x^2 - 5xy - 2y^3\right] = \frac{d}{dx}(8)$$

$$\therefore 6x - 5\left(x\frac{dy}{dx} + y\right) - 2\frac{d}{dx}(y^3) = 0$$

(Product Rule)

$$\therefore 6x - 5x\frac{dy}{dx} - 5y - 6y^2\frac{dy}{dx} = 0$$

(Chain Rule)

$$\therefore 6x - 5y = 5x\frac{dy}{dx} + 6y^2\frac{dy}{dx}$$

$$\therefore 6x - 5y = (5x + 6y^2)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{6x-5y}{5x+6y^2}.$$

(b) $y = \dfrac{(7x^2-2)^{3/7}}{(x^4-4x+7)^{1/4}}$

$$\therefore \ln y = \ln\left[\frac{(7x^2-2)^{3/7}}{(x^4-4x+7)^{1/4}}\right]$$

$$= \ln\left[(7x^2-2)^{3/7}\right] - \ln\left[(x^4-4x+7)^{1/4}\right]$$

$$= \frac{3}{7}\ln(7x^2-2) - \frac{1}{4}\ln(x^4-4x+7)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{3}{7}\left(\frac{1}{7x^2-2}\right)(14x) - \frac{1}{4}\left(\frac{1}{x^4-4x+7}\right)(4x^3-4)$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{6x}{7x^2-2} - \frac{x^3-1}{x^4-4x+7}$$

$$\therefore \frac{dy}{dx} = y\left[\frac{6x}{7x^2-2} - \frac{x^3-1}{x^4-4x+7}\right]$$

(c) (i) $y = x^2\tan^{-1}x$

$\therefore$ By the Product Rule,

$$\frac{dy}{dx} = x^2\left(\frac{1}{x^2+1}\right) + 2x\tan^{-1}x$$

$$= \frac{x^2}{x^2+1} + 2x\tan^{-1}x.$$

(ii) $y = \cosh(3x^2+4)$

Let $u = 3x^2 + 4 \quad \therefore y = \cosh u$

$$\therefore \frac{du}{dx} = 6x \text{ and } \frac{dy}{du} = \sinh u$$

$$\therefore \frac{dy}{dx} = 6x\sinh u \quad \text{(Chain Rule)}$$

$$= 6x\sinh(3x^2+4)$$

2.(c)(iii) $y=\int_0^x \frac{e^{5t}-2t}{4+\sin t}\,dt$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\int_0^x \frac{e^{5t}-2t}{4+\sin t}\,dt\right]$

$= \frac{e^{5x}-2x}{4+\sin x}$.

3.(a)(i) $\cosh x = \frac{1}{2}(e^x+e^{-x})$

$\sinh x = \frac{1}{2}(e^x-e^{-x})$.

(ii) To Prove:

$\cosh(\ln x)+\sinh(\ln x)=x$

Proof: $L.S. = \cosh(\ln x)+\sinh(\ln x)$

$=\frac{1}{2}(e^{\ln x}+e^{-\ln x})+\frac{1}{2}(e^{\ln x}-e^{-\ln x})$

$=\frac{1}{2}(e^{\ln x}+e^{-\ln x}+e^{\ln x}-e^{-\ln x})$

$=\frac{1}{2}(2e^{\ln x})$

$=e^{\ln x}$

$=x$

$=$ R.S, as required.

(b) $y=f(x)=2x^3-6x^2+7$

for $-2 \leq x \leq 3$.

$f(x)$ is continuous and differentiable for $-2\leq x\leq 3$.

$f'(x)=6x^2-12x$

$=6x(x-2)$

$\therefore f'(x)=0$ when

$x=0$ and $x=2$.

$\therefore$ consider 4 points:

$x=-2$; $x=0$; $x=2$; $x=3$.

$\therefore f(-2)=-16-24+7=-33$.

$f(0)=7$.

$f(2)=16-24+7=-1$

$f(3)=54-54+7=7$.

$\therefore$ The absolute maximum is 7 (when $x=0$ and $x=3$).

The absolute minimum is $-33$ (when $x=-2$).

(c) $y=f(x)=\frac{6(x^2-1)}{(x-3)^2}$

(i) Domain: all $x$, except $x=3$.

(ii) Vertical asymptote:

As $x\to 3$, $y\to \frac{48}{0}$ (infinite)

$\therefore$ Vertical asymptote at $x=3$.

(iii) Symmetry: $f(x)=\frac{6(x^2-1)}{(x-3)^2}$

$\therefore f(-x)=\frac{6[(-x)^2-1]}{(-x-3)^2}$

$=\frac{6(x^2-1)}{(-x-3)^2} \neq \begin{cases} f(x) \\ -f(x)\end{cases}$

$\therefore f(x)$ is neither even nor odd.

(iv) Intercepts:

When $x=0$, $y=\frac{-6}{9}=-\frac{2}{3}$.

When $y=0$, $6(x^2-1)=0$

$\therefore x^2=1$

$\therefore x=-1$ and $x=1$.

(v) As $x\to\pm\infty$:

$y=\frac{6(x^2-1)}{(x-3)^2}\sim\frac{6x^2}{x^2}=6$.

Any crossings of $y=6$ occur when $\frac{6(x^2-1)}{(x-3)^2}=6$

$\therefore 6(x^2-1)=6(x-3)^2$

$\therefore x^2-1=(x-3)^2$

$\therefore x^2-1=x^2-6x+9$

$\therefore -1=-6x+9$

$\therefore -10=-6x$

$\therefore x=\frac{10}{6}=\frac{5}{3}$.

(vi) Sign of $y$: As $(x-3)^2\geq 0$, the sign of $y$ is the sign of $x^2-1$.

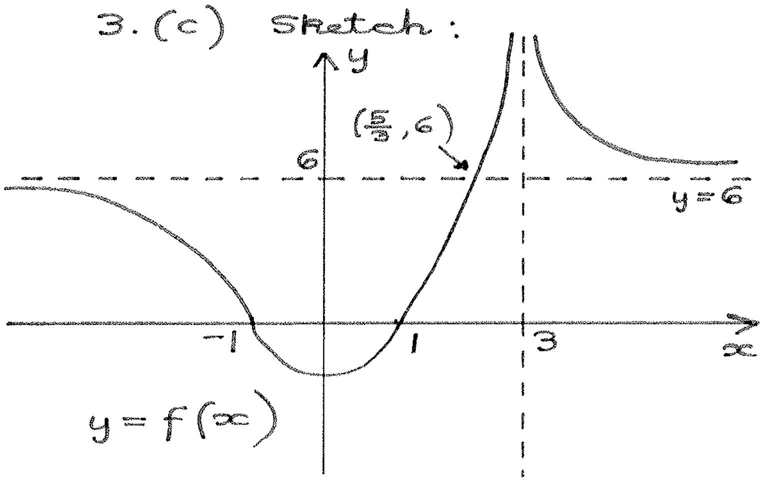
$\therefore y>0$ when $x^2-1>0$

$\therefore x^2>1$

$\therefore x<-1$ and $x>1$.

So, $y<0$ when $-1<x<1$.

3. (c) Sketch:




4. (a) (i) $I = \int \frac{6x^2}{(2x^3-5)^2}\, dx$

Let $u = 2x^3 - 5$

$\therefore \frac{du}{dx} = 6x^2$

$\therefore du = 6x^2\, dx$

and $(2x^3-5)^2 = u^2$

$\therefore I = \int \frac{1}{u^2}\, du = \int u^{-2} du$

$= \frac{u^{-1}}{-1} + C$

$= \frac{-1}{u} + C$

$\therefore I = \frac{-1}{2x^3-5} + C.$

(ii) $I = \int (5x+1) e^{5x}\, dx$

Let $u = 5x+1$ ; $\frac{dv}{dx} = e^{5x}$,

$\therefore \frac{du}{dx} = 5$ ; $v = \frac{1}{5} e^{5x}$

Integrating by parts,

$I = (5x+1)(\frac{1}{5} e^{5x})$

$- \int 5\left(\frac{1}{5} e^{5x}\right) dx$

$= \frac{1}{5}(5x+1)e^{5x} - \int e^{5x}\, dx$

$= \frac{1}{5}(5x+1)e^{5x} - \frac{1}{5} e^{5x} + C$

$= \frac{1}{5}(5x+1-1)e^{5x} + C$

$= \frac{1}{5}(5x)e^{5x} + C$

$\therefore I = x e^{5x} + C.$

(iii) $I = \int \cos x \sqrt{3 + 2\sin x}\, dx$

Let $u = 3 + 2\sin x$

$\therefore \frac{du}{dx} = 2\cos x$

$\therefore du = 2\cos x\, dx$

$\therefore \frac{1}{2} du = \cos x\, dx$

and $\sqrt{3+2\sin x} = \sqrt{u}$

$\therefore I = \int (\sqrt{u})(\frac{1}{2} du)$

$= \frac{1}{2} \int u^{1/2} du$

$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$

$= \frac{1}{3} u^{3/2} + C$

$\therefore I = \frac{1}{3}(3 + 2\sin x)^{3/2} + C.$

(b) $I = \int_0^3 \frac{1}{\sqrt{16+x^2}}\, dx$

$= \left[\ln\{x + \sqrt{16+x^2}\}\right]_0^3$

(S.I. 6 ; a = 4)

$= \ln\{3 + \sqrt{16+9}\} - \ln\{0 + \sqrt{16}\}$

$= \ln(3+5) - \ln 4$

$= \ln 8 - \ln 4$

$= \ln\left(\frac{8}{4}\right) = \ln 2.$

(c) $I = \int \frac{x-8}{x^2-6x+8}\, dx$

As $x^2 - 6x + 8 = (x-2)(x-4)$

$\frac{x-8}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$

$\therefore x - 8 = A(x-4) + B(x-2)$

When $x = 2$, $-6 = -2A$

$\therefore A = 3.$

When $x = 4$, $-4 = 2B$

$\therefore B = -2.$

$\therefore \frac{x-8}{(x-2)(x-4)} = \frac{3}{x-2} - \frac{2}{x-4}$

$\therefore I = 3\ln|x-2| - 2\ln|x-4| + C.$

5. (a) (i) $I = \int_0^2 \frac{3x^2}{x^6+64}\,dx$

Let $u = x^3$

$\therefore \frac{du}{dx} = 3x^2$

$\therefore du = 3x^2\,dx$

and $x^6 + 64 = u^2 + 64$.

<u>Terminals</u>: $x = 0 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 8$.

$\therefore I = \int_0^8 \frac{1}{u^2+64}\,du$,

a standard integral.

(ii) $I = \frac{1}{8}\left[\tan^{-1}\left(\frac{u}{8}\right)\right]_0^8$

(S.I. 3 ; $a = 8$)

$\therefore I = \frac{1}{8}\left(\tan^{-1} 1 - \tan^{-1} 0\right)$

$= \frac{1}{8}\left(\frac{\pi}{4} - 0\right)$

$\therefore I = \frac{\pi}{32}$.

(b) $y = 4x - x^2$

$= x(4 - x)$

is a concave down parabola.

When $y = 0$, $x = 0$ and $x = 4$.

$y = 2x - 3$ is a line.

When $y = 0$, $x = \frac{3}{2}$.

Intersections occur when

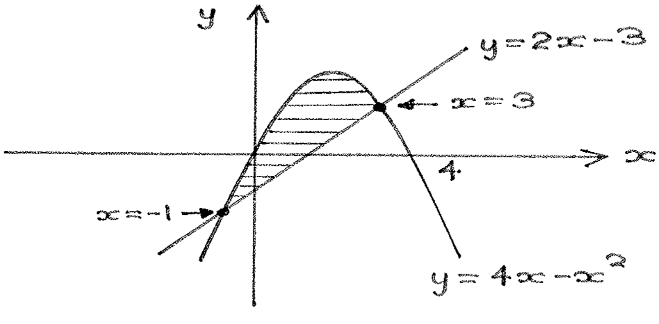
$2x - 3 = 4x - x^2$

$\therefore x^2 - 4x + 2x - 3 = 0$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x+1)(x-3) = 0$

$\therefore x = -1$ and $x = 3$.




From the sketch, the top of the area is $y = 4x - x^2$, and the bottom is $y = 2x - 3$.

$\therefore$ The area $A$ is given by

$A = \int_{-1}^{3} \left(4x - x^2 - 2x + 3\right) dx$

$= \int_{-1}^{3} \left(2x - x^2 + 3\right) dx$

$= \left[x^2 - \frac{x^3}{3} + 3x\right]_{-1}^{3}$

$= \left[9 - \frac{27}{3} + 9\right] - \left[1 + \frac{1}{3} - 3\right]$

$= 9 - 9 + 9 - 1 - \frac{1}{3} + 3$

$= 11 - \frac{1}{3} = \frac{33-1}{3}$

$\therefore$ Area $= \frac{32}{3}$.

(c) $y = (x^2 - 1)\sqrt{x}$

is defined for $x \geq 0$ only

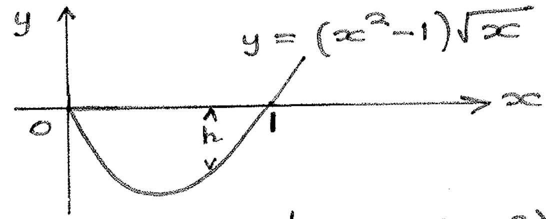
When $y = 0$,

$(x-1)(x+1)\sqrt{x} = 0$

$\therefore x = 1$ or $x = 0$

($x = -1$ is outside the domain)

For $0 < x < 1$, $y < 0$.




From the sketch, $h = (1 - x^2)\sqrt{x}$

$\therefore h^2 = (1 - 2x^2 + x^4)\,x$

$= x - 2x^3 + x^5$.

$\therefore$ The volume $V$ is given by

$V = \pi \int_0^1 \left(x - 2x^3 + x^5\right) dx$

$= \pi \left[\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6}\right]_0^1$

$= \pi \left[\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) - 0\right]$

$\therefore V = \frac{\pi}{6}$.

6. (a) (i) $a_n = \dfrac{2n+(-1)^n}{n}$.

$a_1 = \dfrac{2-1}{1} = 1$

$a_2 = \dfrac{4+1}{2} = \dfrac{5}{2}$

$a_3 = \dfrac{6-1}{3} = \dfrac{5}{3}$

$a_4 = \dfrac{8+1}{4} = \dfrac{9}{4}$.

(ii) In general,

$$a_n = 2 + \frac{(-1)^n}{n}$$

$$= \begin{cases} 2 + \frac{1}{n}, & \text{for } n \text{ even} \\ 2 - \frac{1}{n}, & \text{for } n \text{ odd} \end{cases}$$

As $\lim_{n\to\infty} \left(2+\frac{1}{n}\right) = 2$, and

$\lim_{n\to\infty} \left(2-\frac{1}{n}\right) = 2$,

$\lim_{n\to\infty} a_n = 2$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)4^n}$

$\therefore a_n = \dfrac{(x-3)^n}{(n+1)4^n}$

$\therefore a_{n+1} = \dfrac{(x-3)^{n+1}}{(n+2)4^{n+1}}$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}(n+1)(4^n)}{(n+2)(4^{n+1})(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)(n+1)}{4(n+2)}\right|$$

$$= \frac{|x-3|}{4}\lim_{n\to\infty}\frac{n+1}{n+2}$$

$$= \frac{|x-3|}{4}\lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}$$

$$= \frac{|x-3|}{4}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$$= \frac{|x-3|}{4}.$$

For convergence, $\dfrac{|x-3|}{4} < 1$

$\therefore |x-3| < 4$

$\therefore -4 < x-3 < 4$

$\therefore -1 < x < 7$ is the open interval of convergence.

(c) (i) The first 3 terms of the MacLaurin series for $f(x)$ are given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

$f(x) = (1+x)^{2/3}$ ; $f(0) = 1$.

$f'(x) = \frac{2}{3}(1+x)^{-1/3}$ ; $f'(0) = \frac{2}{3}$.

$f''(x) = \frac{-2}{9}(1+x)^{-4/3}$ ; $f''(0) = \frac{-2}{9}$.

$\therefore f(x) \simeq 1 + \frac{2x}{3} - \left(\frac{1}{2}\right)\left(\frac{2}{9}\right)x^2$

$\therefore (1+x)^{2/3} \simeq 1 + \frac{2x}{3} - \frac{x^2}{9}$.

(ii) $(1+x)^{2/3} = (1.06)^{2/3}$

when $x = 0.06$

$$\therefore (1.06)^{2/3} \simeq 1 + \frac{2\times 0.06}{3} - \frac{(0.06)^2}{9}$$

$= 1 + 0.04 - 0.0004$

$\therefore (1.06)^{2/3} \simeq 1.0396$.

(d) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$

for all $x$.

(i) $\dfrac{1-\cos x}{x} = \dfrac{1}{x}\left[1-\cos x\right]$

$$= \frac{1}{x}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\right]$$

$$= \frac{1}{x}\left[1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right]$$

$$= \frac{1}{x}\left[\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right]$$

$$= \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \ldots$$

(ii) $I = \displaystyle\int_0^1 \frac{1-\cos x}{x}\,dx$

$$\simeq \int_0^1 \left(\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!}\right)dx$$

$$= \int_0^1 \left(\frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720}\right)dx$$

$$= \left[\frac{x^2}{4} - \frac{x^4}{96} + \frac{x^6}{4320}\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{1}{96} + \frac{1}{4320}\right) - 0$$

$$= \frac{1080 - 45 + 1}{4320} = \frac{1036}{4320}$$

$\therefore I \simeq 0.2398$.

7. (a) $z = 5-3i$ , $w = 3+4i$

(i) $\overline{w} = 3-4i$

$\therefore 2z-\overline{w} = 2(5-3i)-(3-4i)$

$= 10-6i-3+4i$

$= 7-2i$.

(ii) $zw = (5-3i)(3+4i)$

$= 5(3+4i)-3i(3+4i)$

$= 15+20i-9i-12i^2$

$= 15+11i+12$

$= 27+11i$ .

(iii) $\dfrac{z}{w} = \dfrac{5-3i}{3+4i}$

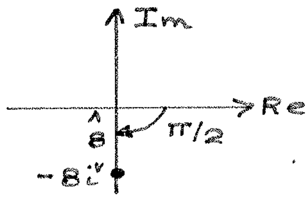
$= \dfrac{(5-3i)(3-4i)}{(3+4i)(3-4i)}$

$= \dfrac{15-20i-9i+12i^2}{3^2+4^2}$

$= \dfrac{15-29i-12}{9+16}$

$= \dfrac{3-29i}{25}$ .

(b) $z^3 = -8i$




Let $z = r \operatorname{cis} \theta$

$\therefore z^3 = r^3 \operatorname{cis} 3\theta$

From diagram,

$-8i = 8 \operatorname{cis}\left(\frac{-\pi}{2} + 2k\pi\right)$

$\therefore r^3 \operatorname{cis} 3\theta = 8 \operatorname{cis}\left(\frac{-\pi}{2} + 2k\pi\right)$

$\therefore r^3 = 8$ and $3\theta = \frac{-\pi}{2} + 2k\pi$

$\therefore r = 2$ and

$3\theta = \dfrac{-\pi + 4k\pi}{2} = \dfrac{\pi(4k-1)}{2}$

$\therefore \theta = \dfrac{\pi(4k-1)}{6}$

Choosing $k = 0, 1, 2$ gives

$\theta = \frac{-\pi}{6}$, $\theta = \frac{3\pi}{6} = \frac{\pi}{2}$,

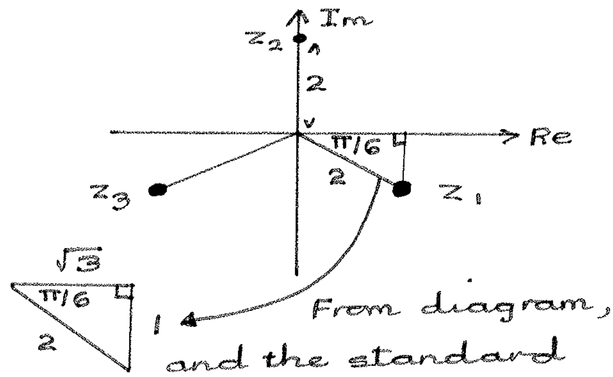
and $\theta = \frac{7\pi}{6}$.

$\therefore$ The 3 solutions are:

$z_1 = 2 \operatorname{cis}\left(\frac{-\pi}{6}\right)$;

$z_2 = 2 \operatorname{cis} \frac{\pi}{2}$ ;

$z_3 = 2 \operatorname{cis} \frac{7\pi}{6}$ .




From diagram, and the standard triangle,

$z_1 = \sqrt{3} - i$ ;

$z_2 = 2i$ ;

$z_3 = -\sqrt{3} - i$ (symmetry)

(c) $\underset{\sim}{A} = 4\underset{\sim}{i} - \underset{\sim}{j} + 8\underset{\sim}{k}$, and

$\underset{\sim}{B} = 2\underset{\sim}{i} + 4\underset{\sim}{j} + 4\underset{\sim}{k}$.

(i) $|\underset{\sim}{A}| = \sqrt{4^2+(-1)^2+8^2}$

$= \sqrt{16+1+64} = \sqrt{81}$

$\therefore |\underset{\sim}{A}| = 9$

$|\underset{\sim}{B}| = \sqrt{2^2+4^2+4^2}$

$= \sqrt{4+16+16} = \sqrt{36}$

$\therefore |\underset{\sim}{B}| = 6$.

(ii) $\underset{\sim}{A} \cdot \underset{\sim}{B} = (4)(2)+(-1)(4)+(8)(4)$

$= 8-4+32$

$\therefore \underset{\sim}{A} \cdot \underset{\sim}{B} = 36$.

(iii) $\cos\theta = \dfrac{\underset{\sim}{A} \cdot \underset{\sim}{B}}{|\underset{\sim}{A}||\underset{\sim}{B}|}$

$= \dfrac{36}{(9)(6)} = \dfrac{6}{9} = \dfrac{2}{3}$

$\therefore \cos\theta = \dfrac{2}{3}$.

(iv) $\underset{\sim}{A} \times \underset{\sim}{B} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 4 & -1 & 8 \\ 2 & 4 & 4 \end{vmatrix}$

$= \underset{\sim}{i} \begin{vmatrix} -1 & 8 \\ 4 & 4 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} 4 & 8 \\ 2 & 4 \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} 4 & -1 \\ 2 & 4 \end{vmatrix}$

$= \underset{\sim}{i}(-4-32) - \underset{\sim}{j}(16-16) + \underset{\sim}{k}(16+2)$

$= -36\underset{\sim}{i} + 18\underset{\sim}{k}$

$\therefore \underset{\sim}{A} \times \underset{\sim}{B} = -36\underset{\sim}{i} + 18\underset{\sim}{k}$

$\left(= 18\left(-2\underset{\sim}{i} + \underset{\sim}{k}\right)\right)$.

8. (a) (i) Given $P(1,-2,3)$, $Q(4,0,1)$ and $R(-2,0,3)$, two vectors in the plane are

$\overrightarrow{PQ} = (4-1)\underset{\sim}{i} + (0+2)\underset{\sim}{j} + (1-3)\underset{\sim}{k}$

$\therefore \overrightarrow{PQ} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$,

and $\overrightarrow{PR} = (-2-1)\underset{\sim}{i} + (0+2)\underset{\sim}{j} + (3-3)\underset{\sim}{k}$

$\therefore \overrightarrow{PR} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$.

$\therefore$ A vector perpendicular to the plane, $\underset{\sim}{N}$, is given by

$\underset{\sim}{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$\therefore \underset{\sim}{N} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 2 & -2 \\ -3 & 2 & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} 2 & -2 \\ 2 & 0 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 3 & -2 \\ -3 & 0 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 3 & 2 \\ -3 & 2 \end{vmatrix}$$

$= \underset{\sim}{i}(0+4) - \underset{\sim}{j}(0-6) + \underset{\sim}{k}(6+6)$

$\therefore \underset{\sim}{N} = 4\underset{\sim}{i} + 6\underset{\sim}{j} + 12\underset{\sim}{k}$ is a vector perpendicular to the plane (as is any non-zero multiple).

(ii) The area of the triangle $PQR$ is given by $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$= \frac{1}{2}|4\underset{\sim}{i} + 6\underset{\sim}{j} + 12\underset{\sim}{k}|$

$= |2\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k}|$

$= \sqrt{2^2+3^2+6^2} = \sqrt{4+9+36}$

$= \sqrt{49} = 7$.

(iii) The plane through $P$, $Q$, and $R$ has equation

$4x + 6y + 12z = D$.

(as $\underset{\sim}{N} = 4\underset{\sim}{i} + 6\underset{\sim}{j} + 12\underset{\sim}{k}$ is a normal to the plane).

Since $Q(4,0,1)$ is in the plane, $4 \times 4 + 0 + 12 \times 1 = D$

$\therefore D = 16 + 12 = 28$.

$\therefore$ The equation to the plane is

$4x + 6y + 12z = 28$

i.e. $2x + 3y + 6z = 14$.

(b) Given $P(3,-1,2)$, and $Q(6,0,1)$, a vector parallel to the line is $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = (6-3)\underset{\sim}{i} + (0+1)\underset{\sim}{j} + (1-2)\underset{\sim}{k}$

$\therefore \overrightarrow{PQ} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

Using $P$ as the point on the line, the parametric equations are:

$x = 3+3t$; $y = -1+t$; $z = 2-t$.

(c) Substituting the equations of the line

$x = 4-t$, $y = -3+2t$, $z = 1+6t$

into the equation of the plane

$4x - 3y + 2z = 19$ gives

$4(4-t) - 3(-3+2t) + 2(1+6t) = 19$

$\therefore 16 - 4t + 9 - 6t + 2 + 12t = 19$

$\therefore 2t + 27 = 19$

$\therefore 2t = -8 \quad \therefore t = -4$.

When $t = -4$,

$x = 4+4 = 8$, $y = -3-8 = -11$,

and $z = 1-24 = -23$.

$\therefore$ The point of intersection is $(8, -11, -23)$.

(d) Given $\underset{\sim}{A} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{B} = 3\underset{\sim}{i} - \underset{\sim}{j}$, and $\underset{\sim}{C} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$,

$$\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = \begin{vmatrix} 2 & 1 & -2 \\ 3 & -1 & 0 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix}$$

$= 2(-1-0) - 1(3-0) - 2(9+4)$

$= -2 - 3 - 26 = -31$

$\therefore \underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C}) = -31$.

The volume, $V$, is given by

$V = |\underset{\sim}{A} \cdot (\underset{\sim}{B} \times \underset{\sim}{C})| = |-31|$

$\therefore V = 31$.