DEAKIN UNIVERSITY

FACULTY OF SCIENCE & TECHNOLOGY

School of Information Technology

2004 EXAMINATION PAPER

UNIT CODE: SCM124

UNIT NAME: Introduction to Mathematical Modelling

EXAMINATION: November

WRITING TIME: 3 hours

READING TIME: 15 minutes

CONDITIONS: Closed book exam. Calculators are allowed.

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Attempt as many questions as possible.
A formula sheet is attached.

1. (a) For the function $y = f(x) = |2x - 1|$:

   (i) write the domain and range of the function and sketch it;

   (ii) show, using algebra, that $f(x)$ is not one-to-one;

   (iii) find a restriction of the domain such that the function is one-to-one.

   (b) For $y = f(x) = \sqrt{6x - 8}, \qquad x \geq \frac{4}{3}$:

   (i) find $f^{-1}(x)$;

   (ii) sketch $f(x)$ and $f^{-1}(x)$ on the same set of axes and label all intersections of the two functions.

   (c) Evaluate the following limits

   (i) $\lim\limits_{x \to \infty} \dfrac{3x^2 - x + 2}{4x^2 + 5}$ (ii) $\lim\limits_{x \to 4} \dfrac{x^2 - x - 12}{x^2 - 4x}$

   (iii) $\lim\limits_{x \to 1} \dfrac{3x^5 + 2x^3 - x - 4}{x^2 - 1}$ (iv) $\lim\limits_{x \to 0} \dfrac{e^{2x} - 3 + 2\cos x}{5x}$.

[8+7+12 = 27Marks]

2. (a) Find $\frac{dy}{dx}$ in the following cases:

   (i) $y = \dfrac{2x + 3}{3x - 4}$ (ii) $y = (x\sin x + \cos x)^4$

   (iii) $y = (9x^2 - 6x + 2)e^{3x}$ (iv) $5x^3 - 7xy - y^2 = 1$.

   (b) Use logarithmic differentiation to find $\frac{dy}{dx}$ if $y = \dfrac{(5x - 2)^{1/5}}{(x^3 - 3x + 4)^{2/3}}$

---

2. (c) Find $\frac{dy}{dx}$ in the following cases:

(i) $y = \sin^{-1}(x^2)$ (ii) $y = x^3 \cosh x$

(iii) $y = \int_0^x \frac{\ln(t+1)}{3+\cos t} dt$.

[12+5+7=24 Marks]

3. (a) (i) State the definitions of $\cosh x$ and $\sinh x$.

(ii) Using the definitions in (i), prove that $\ln(\cosh x + \sinh x) = x$.

(b) Find the absolute maximum and absolute minimum of $y = f(x) = x^3 - 12x + 1$ for $-3 \le x \le 5$.

(c) Sketch $y = f(x) = \frac{4(x^2-9)}{(x-6)^2}$ after examining:

(i) domain; (ii) vertical asymptotes;
(iii) symmetry; (iv) intercepts;
(v) behaviour as $x \to \pm\infty$; (vi) sign of y.

[5+5+11=21 Marks]

4. (a) Find

(i) $I = \int \frac{2x}{3x^2-7} dx$

(ii) $I = \int (2x+1)e^{2x}\, dx$

(iii) $I = \int \frac{\cos x}{\sqrt{4+\sin x}} dx$.

(b) Use a standard integral to evaluate, and express in terms of π

$$I = \int_0^2 \frac{1}{\sqrt{16-x^2}} dx.$$

---

4. (c) Use partial fractions to find $I = \int \frac{x+2}{x^2 - 5x + 4} dx$.

[12+4+5 = 21 marks]

5. (a) (i) Convert $I = \int_0^2 \frac{2x}{\sqrt{x^4 + 9}} dx$ to a standard integral by making the substitution $u = x^2$.

(ii) Use the standard integral obtained in (i) to evaluate *I*, and express in terms of the natural logarithm.

(b) Sketch, and find the finite area bounded by the curve $y = x^2 - 2x + 1$, and the line $y = x + 1$.

(c) Find the volume formed when the area below the curve $y = x(1 - \sqrt{x})$, and above the x axis, is rotated about the x axis.

[6+8+6=20 Marks]

6. (a) (i) Find the first four terms of the sequence $\{a_n\}$ with n^{th} term

$$a_n = \frac{3 + (-1)^n}{3n}.$$

(ii) Find $\lim_{n \to \infty} a_n$ (if it exists).

(b) Find the open interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{(n+6)3^n}.$$

(c) (i) Derive the first <u>three</u> terms of the MacLaurin series for

$$f(x) = (1+x)^{-3/2}.$$

(ii) Hence approximate $(1.04)^{-3/2}$.

---

6. (d) Given that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots$ for $-\infty < x < \infty$:

(i) write down a power series for $\frac{e^{-x} - 1}{x}$;

(ii) use the first three non-zero terms of the series for $\frac{e^{-x} - 1}{x}$ to approximate

$$I = \int_0^1 \frac{e^{-x} - 1}{x} dx.$$

[5+5+6+7 = 23 Marks]

7. (a) Given that $z = 5 - 4i$, and $w = 2 + i$, simplify

(i) $\bar{z} - 2w$

(ii) zw

(iii) $\frac{z}{w}$.

(b) Solve $z^3 = -8$, and express all solutions in Cartesian form.

(c) Given $\underset{\sim}{A} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$, and $\underset{\sim}{B} = 4\underset{\sim}{i} + 8\underset{\sim}{j} - \underset{\sim}{k}$, find:

(i) $|\underset{\sim}{A}|$ and $|\underset{\sim}{B}|$;

(ii) $\underset{\sim}{A} \cdot \underset{\sim}{B}$;

(iii) the cosine of the angle between $\underset{\sim}{A}$ and $\underset{\sim}{B}$;

(iv) $\underset{\sim}{A} \times \underset{\sim}{B}$.

[6+7+8=21 Marks]

8. (a) (i) Find a vector perpendicular to the plane containing the points P(1,-2,5), Q(3,0,8), and R(1,-4,8).

(ii) Hence find the area of the triangle PQR.

(iii) Find the equation of the plane containing the points P, Q, and R.

(b) Find parametric equations of the line through the points P(5,-1,7) and Q(3,0,4).

(c) Find the point at which the line

$$x = 4 - 3t,\ y = -5 + t,\ z = 1 + 2t$$

intersects the plane $4x - 3y + 5z = 1$.

(d) Find the volume of the box whose edges are determined by the vectors

$\underset{\sim}{A} = 3\underset{\sim}{i} + \underset{\sim}{j} + 7\underset{\sim}{k}$, $\underset{\sim}{B} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, and $\underset{\sim}{C} = \underset{\sim}{i} - 6\underset{\sim}{j}$.

[8+4+5+4=21 Marks]

List of Standard Integrals and Trigonometric Formulae

Standard Integrals (+C omitted)

Function		Integral
1	$1/(a^2 - x^2)$	$\frac{1}{a}\tanh^{-1}(x/a)$ or $\frac{1}{2a}\ln\frac{a+x}{a-x}$ if $\lvert x\rvert < a$
2	$1/(x^2 - a^2)$	$-\frac{1}{a}\coth^{-1}(x/a)$ or $\frac{1}{2a}\ln\frac{x-a}{x+a}$ if $\lvert x\rvert > a$
3	$1/(x^2 + a^2)$	$\frac{1}{a}\tan^{-1}(x/a)$
4	$1/\sqrt{a^2 - x^2}$	$\sin^{-1}(x/a)$
5	$1/\sqrt{x^2 - a^2}$	$\cosh^{-1}(x/a)$ or $\ln\{x + \sqrt{x^2 - a^2}\}$ if $x > a$
		$-\cosh^{-1}(-x/a)$ or $\ln\{-x + \sqrt{x^2 - a^2}\}$ if $x < -a$
6	$1/\sqrt{x^2 + a^2}$	$\sinh^{-1}(x/a)$ or $\ln\{x + \sqrt{x^2 + a^2}\}$
7	$\sqrt{a^2 - x^2}$	$\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}(x/a)$
8	$\sqrt{x^2 - a^2}$	$\frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\cosh^{-1}(x/a)$ if $x \geq a$
		$\frac{1}{2}x\sqrt{x^2 - a^2} + \frac{1}{2}a^2\cosh^{-1}(-x/a)$ if $x \leq -a$
9	$\sqrt{x^2 + a^2}$	$\frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2\sinh^{-1}(x/a)$
10	$e^{ax}\sin bx$	$\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$
11	$e^{ax}\cos bx$	$\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$

Reduction Formulae

12 $$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

$$\text{or } -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx$$

13 $$\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

14 $$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

Trigonometric Formulae

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\sin x\cos x = \frac{1}{2}\sin 2x$$